Below is Giuseppe Peano's *Arithmetices principia* as first published[1], i.e. an "*Arithmetices principia, nova methodo exposita*"[2], which appeared translated to English[3] in 1967 as "*The principles of arithmetic, presented by a new method*"[4], as well as in 1973[5]. This present document[6] is the only (to my knowledge) side-by-side Latin-English translation of the Latin original. The mathematical notation (in the English, right, column) got updated to currently canonically used or easy-to-decrypt symbols in the international and/or English mathematical community, which is also a feature currently unseen in any reprints. In the middle of each dotted horizontal line is the number of the page where the page below in the original Latin edition of 1889. Parts irrelevant for the mathematical content and/or current notation got greyed.

I

Arithmetices principia
Nova methodo exposita
a
Ioseph Peano
in R. Academia militari professore
Analysis infinitorum in R. Taurinensi Athenaeo docente.

Labor et honor

Augustae Taurinorum
Ediderunt Fratres Bocca
Regis bibliopolae

Romae, Via del Corso, 216-217. — Florentiae, Via Cerretani, 8.

1889

The principles of arithmetic
Presented by a new method
by
Giuseppe Peano
professor at the Royal Military Academy
teaching Analysis of the infinite at the Royal Turin Athenaeum.

Work and honor

At Turin
Published by Libreria Bocca

At Rome, Via del Corso, 216-217. — At Florence, Via Cerretani, 8.

1889

II

Iuribus servatis.

Augustae Taurinorum - Typis Vincentii Bona.

Reserving rights.

At Turin, printed by Vincent Bona.

III

Praefatio

Quaestiones, quae ad mathematicae fundamenta pertinent, etsi hisce temporibus a multis tractatae, satisfacienti solutione et adhuc carent. Hic difficultas maxime ex sermonis ambiguitate oritur.

Quare summi interest verba ipsa, quibus utimur, attente perpendere. Hoc examen mihi proposui, atque mei studii resultatus, et arithmeticae applicationes in hoc scripto expono.

Ideas omnes quae in arithmeticae principiis occurrunt, signis indicavi, ita ut quaelibet propositio his tantum signis enuncietur.

Signa aut ad logicam pertinent, aut propria ad arithmeticam. Logicae signa quae hic occurrunt, sunt numero ad decem, quamvis non omnia necessaria. Horum signorum usus et proprietates nonnullae in prima parte communi sermone explicantur. Ipsorum theoriam fusius hic exponere nolui. Arithmeticae signa, ubi occurrunt, explicantur.

His notationibus quaelibet propositio formam assumit atque praecisionem, qua in algebra aequationes gaudent, et a propositionibus ita scriptis aliae deducuntur, idque processibus qui aequationum resolutioni assimilantur. Hoc caput totius scripti.

Signis igitur, quibus propositiones arithmeticae scribere possim, in earum tractatione methodo usus sum, quam, et in aliis studiis sequendam, breviter exponam.

Ex arithmeticae signis quae cum caeteris, una cum logicae signis exprimere licet, ideas significant quas definire possumus. Ita omnia definivi signa, si quatuor excipias, quae in explicationibus §1 continentur. Si, ut puto, haec ulterius reduci nequeunt, ideas ipsis expressas, per ideas prius notas supponendas, definire non licet.

IV

Propositiones, quae logicae operationibus a caeteris deducuntur, sunt theoremata; quae vero non, axiomata vocavi. Axiomata hic sunt novem (§1), et signorum, quae definitione carent, proprietates fundamentales exprimunt.

In §1-6 numerorum proprietates communes demonstravi; brevitatis causa, demonstrationes praecedentibus similes omisi. Demonstrationum formam communem immutare oportuit, ut logicae signis exprimi possint; haec transformatio interdum difficilior est, tamen inde demonstrationis natura clarissime patet.

In sequentibus § varia tractavi, ut huius methodi potentia magis videatur.

In §7 nonnulla theoremata, quae ad numerorum theoriam pertinent, continentur. In §8 et 9 rationalium et irrationalium definitiones inveniuntur.

Denique, in §10, theoremata exposui nonnulla, quae nova esse puto, ad entium theoriam pertinentia, quae cl.[mus] Cantor *Punktmenge* (*ensemble de points*) vocavit.

In hoc scripto aliorum studiis usus sum. Logicae notationes et propositiones quae in numeris II, III et IV continentur, si nonnullas excipias, multorum opus exhibent, atque Boole praecipue.

V

Signum ε, quod cum signo Ↄ non est confundendum, inversionis in logica applicationes, et pauca alia conventiones, ut ad quaslibet propositiones exprimendas pervenirem.

In arithmeticae demonstrationibus usus sum libro: H. Grassmann, *Lehrbuch der Arithmetik*, Berlin 1861.

Utilissimum quoque mihi fuit recens scriptum: R. Dedekind, *Was sind und was sollen die Zahlen*, Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur.

Hic meus libellus ad novae methodi specimen habendus est. Hisce notationibus innumeras alias propositiones, ut quae ad rationales et irrationales pertinent, enunciare et demonstrare possumus. Sed, ut aliae theoriae tractentur, nova signa, quae nova indicant entia, introducere oportet. Puto vero his tantum logicae signis propositiones cuiuslibet scientiae exprimi posse, dummodo adiungantur signa quae entia huius scientiae repraesentant.

Preface

Questions pertaining to the foundations of mathematics, although treated by many these days, still lack a satisfactory solution. The difficulty arises principally from the ambiguity of ordinary language.

For this reason it is of the greatest concern to consider attentively the words we use. I resolved to do this, and am presenting in this paper the results of my study with applications to arithmetic.

I have indicated by signs all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these signs.

The signs pertain either to logic or to arithmetic. The signs of logic that occur here are about ten in number, although not all are necessary. The use of these signs and several of their properties are explained in ordinary language in the first part. I did not wish to present their theory more fully here. The signs of arithmetic are explained as they occur.

With this notation every proposition assumes the form and precision equations enjoy in algebra, and from propositions so written others may be deduced, by a process which resembles the solution of algebraic equations. That is the chief reason for writing this paper.

Having made up the signs with which I can write arithmetical propositions, in treating them I have used a method which, because it is to be followed in later studies, I shall present briefly.

Those arithmetical signs which may be expressed by using others along with signs of logic represent the ideas we can define. Thus I have defined every sign, if you except the four which are contained in the explanations of §1. If, as I believe, these cannot be reduced further, then the ideas expressed by them may not be defined by ideas already supposed to be known.

Propositions which are deduced from others by the operations of logic are theorems; those for which this is not true I have called axioms. There are nine axioms here (§1), and they express fundamental properties of the undefined signs.

In §1-6 I have proved the ordinary properties of numbers; for the sake of brevity, I have omitted proofs which are similar to preceding ones. The ordinary form of proofs has had to be altered in order that they may be expressed with the signs of logic. This transformation is sometimes rather difficult but the nature of the proof then becomes quite clear.

In the following sections I have treated various things so that the power of the method is better seen.

In §7 are several theorems pertaining to the theory of numbers. In §8 and 9 are found the definitions of rational and irrational.

Finally, in §10 I have given several theorems, which I believe to be new, pertaining to the theory of those entities which Professor Cantor has called *Punktmenge* (*ensemble de points*).

In this paper I have used the research of others. The notations and propositions of logic which are contained in numbers II, III, and IV, with some exceptions, represent the work of many, among them Boole especially.

The sign ∈, which must not be confused with the sign ⊂, applications of the inverse in logic, and a few other conventions, I have adopted so that I could express any proposition whatever.

In the proofs of arithmetic I used the book H. Grassmann, *Lehrbuch der Arithmetik* (Berlin 1861).

Also quite useful to me was the recent work by R. Dedekind, *Was sind und was sollen die Zahlen* (Braunschweig, 1888), in which questions pertaining to the foundations of numbers are acutely examined.

My booklet should be taken as a sample of this new method. With these notations we can state and prove innumerable other propositions, such as those which pertain to rationals and irrationals. But in order to treat other theories, it is necessary to adopt new signs to indicate new entities. I believe, however, that with only these signs of logic the propositions of any science can be expressed, so long as the signs which represent the entities of the science are added.

VI

Signorum tabula

Signum	Significatio	Pag.
Logicae signa		
P	propositio	VII
K	classis	X
∩	et	VII, X
∪	vel	VIII, X
−	non	VIII, X
Λ	absurdum aut nihil	VIII, XI
Ↄ	deducitur aut continetur	VIII, XI
=	est aequalis	VIII
ε	est	X
[ ]	inversionis signum	XI
ι	qui vel [ε]	XII
Th	Theorema	XVI
Hp	Hypothesis	
Ts	Thesis	
L	Logica	

Arithmeticae signa

Signa 1, 2, ..., =, +, −, ×, ÷ vulgarem habent significationem. Divisionis signum est /.

N	numerus integer positivus	1
R	num. rationalis positivus	12
Q	quantitas, sive numerus realis positivus	16
Np	numerus primus	8
M	maximus	8
m	minimus	8
T	terminus vel fines summus	15
D	dividit	8
∞	est multiplex	8
π	est primus cum	8

Signa composita

−<	non est minor
=∪>	est aequalis aut maior
∋D	divisor
M ∋D	maximus divisor

Table of signs

Sign	Meaning	Pag.
Signs of logic		
proposition	proposition	VII
CLASS	class	X
∧	and	VII, X
∨	or	VIII, X, XI
¬	not	VIII, X
⊥	false or	VIII, XI
∅	nothing	
→	one deduces or	VIII, XI
⊂	is contained in	
=	equals	VIII
∈	is an element of	X
is a	is (a)	
(inversion)	the inverse	XI
such that	such that or	XII
isn't	isn't	
Theorem	Theorem	XVI
Hypothesis	Hypothesis	
Thesis	Thesis	
logic	Logic	

Signs of arithmetic

The signs 1, 2, ..., =, +, −, ×, ÷ have their usual meaning. The sign of division is /.

N	positive integers	1
R	positive rational numbers	12
Q	quantity or positive real numbers	16
P	prime number	8
M	maximum	8
m	minimum	8
greatest bound	terminus or greatest bound	15
divides	divides	8
is divisible	is divisible	8
is prime with	is prime with	8

Composite signs

≮	is not less than
≥	is equal to or greater than
is a divisor	is a divisor
is the greatest divisor	is the greatest divisor

VII

Logicae notationes.

I. De punctuatione.

Litteris $a, b, \ldots x, y, \ldots x', y', \ldots$ entia indicamus indeterminata quaecumque. Entia vero determinata signis, sive litteris $P, K, N, \ldots$ indicamus.

Signa plerumque in eadem linea scribemus. Ut ordo pateat quo ea coniungere oporteat, *parenthesibus* ut in algebra, sive *punctis* . : ∴ :: etc. utimur.

Ut formula punctis divisa, intelligatur, primum signa quae nullo puncto seperantur colligenda sunt, postea quae uno puncto, deinde quae duobus punctis, etc.

Ex. g. sint $a, b, c, \ldots$ signa quaecumque. Tunc $ab \,.\, cd$ significat $(ab)(cd)$; et $ab \,.\, cd : ef \,.\, gh ∴ k$ significat $(((ab)(cd))((ef)(gh)))k$.

Punctuationis signa omittere licet si formulae quae diversa punctuatione existerent eundem habeant sensum; vel si una tantum formula, et ipsa quam scribere volumus, sensum habeat.

Ut ambiguitatis periculum absit, aritmeticae operationum signis . : nunquam utimur.

Parenthesum figura una est ( ); si in eadem formula, parentheses et puncta occurant, primum quae parenthesibus continentur, colligantur.

II. De propositionibus.

Signo P significatur *propositio*.

Signum ∩ legitur *et*. Sint a, b propositiones; tunc $a \cap b$ est simultanea affirmatio propositionum a, b. Brevitatis causa, loco $a \cap b$ vulgo scribemus $a\ b$.

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Signum − legitur *non*. Sit a quaedam P; tunc $-a$ est negatio propositionis a.

Signo ∪ legitur *vel*. Sint a, b propositiones; tunc $a \cup b$ idem est ac $- : -a \,.\, -b$.

[Signo V significatur *verum*, sive *identitas*; sed hoc signo numquam utimur].

Signum Λ significat *falsum*, sive *absurdum*.

[Signum C significat *est consequentia*; ita b C a legitur *b est consequentia propositionis a*. Sed hoc signo nunquam utimur].

Signum Ↄ significat *deducitur*; ita a Ↄ b significat quod b C a. Si propositiones a, b entia indeterminata continent $x, y, \ldots$, scilicet sunt inter ipsa entia conditiones, tunc a $Ↄ_{x,y,\ldots}$ b significat: quaecumque sunt $x, y, \ldots$, a propositione a deducitur b. Si vero ambiguitatis periculum absit, loco $Ↄ_{x,y,\ldots}$, scribemus solum Ↄ.

Signum = significat *est aequalis*. Sint a, b propositiones; tunc $a = b$ idem significat quod a Ↄ b . b Ↄ a; propositio $a =_{x,y,\ldots} b$ idem significat quod a $Ↄ_{x,y,\ldots}$ b . b $Ↄ_{x,y,\ldots}$ a.

III. Logicae propositiones.

Sint $a, b, c, \ldots$ propositiones. Tunc erit:

1. a Ↄ a
2. a Ↄ b . b Ↄ c : Ↄ : a Ↄ c
3. $a = b$. = : a Ↄ b . b Ↄ a.
4. $a = a$
5. $a = b$. = . $b = a$
6. $a = b$. b Ↄ c : Ↄ . a Ↄ c
7. a Ↄ b . $b = c$: Ↄ . a Ↄ c
8. $a = b$. $b = c$: Ↄ . $a = c$
9. $a = b$. Ↄ . a Ↄ b
10. $a = b$. Ↄ . b Ↄ a

11. ab Ↄ a
12. $ab = ba$
13. $a(bc) = (ab)c = abc$

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14. $aa = a$
15. $a = b$. Ↄ . $ac = bc$
16. a Ↄ b . Ↄ . ac Ↄ bc
17. a Ↄ b . c Ↄ d : Ↄ . ac Ↄ bd
18. a Ↄ b . a Ↄ c : = . a Ↄ bc
19. $a = b$. $c = d$: Ↄ . $ac = bd$

20. $-(-a) = a$
21. $a = b$. = . $-a = -b$.
22. a Ↄ b . = . $-b$ Ↄ $-a$ =

23. $a \cup b$. = ∴ − : $-a$. $-b$
24. $-(ab) = (-a) \cup (-b)$
25. $-(a \cup b) = (-a)(-b)$
26. a Ↄ . $a \cup b$
27. $a \cup b = b \cup a$
28. $a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c$
29. $a \cup a = a$
30. $a(b \cup c) = ab \cup ac$
31. $a = b$. Ↄ . $a \cup c = b \cup c$
32. a Ↄ b . Ↄ . $a \cup c$ Ↄ $b \cup c$
33. a Ↄ b . c Ↄ d : Ↄ : $a \cup c$. Ↄ . $b \cup d$
34. b Ↄ a . c Ↄ a : = . $b \cup c$ Ↄ a

35. $a - a = \Lambda$
36. $a\,\Lambda = \Lambda$
37. $a \cup \Lambda = \mathrm{a}$
38. a Ↄ Λ . = . $a = \Lambda$
39. a Ↄ b . = . $a - b = \Lambda$
40. Λ Ↄ a
41. $a \cup b = \Lambda$. = : $a = \Lambda$. $b = \Lambda$

42. a Ↄ . b Ↄ c : = : ab Ↄ c
43. a Ↄ . $b = c$: = . $ab = ac$

X

Sit α quoddam relationis signum (ex. gr. =, Ↄ), ita ut $a\ \alpha\ b$ sit quaedam propositio. Tunc loco − . $a\ \alpha\ b$ scribemus $a - \alpha\ b$; scilicet:

$a - = b$. = : − . $a = b$

$a -$ Ↄ b . = : − . a Ↄ b

Ita signum − = significat *non est aequalis*. Si propositio a indeterminatum continet x, $a -=_x \Lambda$ significat: sunt x quae conditioni a satisfaciunt. Signum − Ↄ significat *non deducitur*.

Similter, si α et β sunt relationis signa, loco $a\ \alpha\ b$, et $a\ \alpha\ b$. ∪ . $a\ \beta\ b$ scribere possumus a . $\alpha\beta$. b et a . $\alpha \cup \beta$. b. Ita, si a et b sunt propositiones, formula a . Ↄ − = . b dicit: ab a *deducitur* b, sed non vice versa.

a . Ↄ − = . b : = : a Ↄ b . b − Ↄ a

Formulae:

a Ↄ b . b Ↄ c . a − Ↄ c : = Λ

$a = b$. $b = c$. $a - = c$: = Λ

a Ↄ b . b Ↄ − = c : Ↄ . a Ↄ − = c

a Ↄ − = b . b Ↄ c : Ↄ . a Ↄ − = c

Sed his notationibus raro utimur.

IV. De classibus.

Signo K significatur *classis*, sive entium aggregatio.

Signum ε significat *est*. Ita a ε b legitur *a est quoddam b*; a ε K significat *a est quaedam classis*; a ε P significat *a est quaedam propositio*.

Loco $-(a\ \varepsilon\ b)$ scribemus $a - \varepsilon\ b$; signum − ε significat *non est*; scilicet:

44. $a - \varepsilon\ b$. = : − . $a\ \varepsilon\ b$

Signum $a, b, c\ \varepsilon\ m$ significat: a, b et c sunt m; scilicet:

45. $a, b, c\ \varepsilon\ m$. = : $a\ \varepsilon\ m$. $b\ \varepsilon\ m$. $c\ \varepsilon\ m$

Sit a classis; tunc $-a$ significatur classis indiviuis constituta quae non sunt a.

46. $a\ \varepsilon$ K . Ↄ : $x\ \varepsilon - a$. = . $x - \varepsilon\ a$

Sint a, b classes; $a \cap b$, sive $a\ b$, est classis individuis constituta

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quae eodem tempore sunt a et b; $a \cup b$ est classis individuis constituta qui sunt a vel b.

47. $a, b\ \varepsilon$ K . Ↄ ∴ $a\ x\ \varepsilon$. $a\ b$: = : $x\ \varepsilon\ a$. $x\ \varepsilon\ b$
48. $a, b\ \varepsilon$ K . Ↄ ∴ $a \cup x\ \varepsilon$. $a \cup b$: = : $x\ \varepsilon\ a$. ∪ . $x\ \varepsilon\ b$

Signum Λ indicat classem quae nullum continet individuum. Ita:

49. $a\ \varepsilon$ K . Ↄ ∴ $a = \Lambda$: = : $x\ \varepsilon\ a$. $=_x \Lambda$

[Signo V, quod classem ex omnibus individuis constitutam, de quibus quaestio est, indieat, non utimur].

Signum Ↄ significat *continetur*. Ita a Ↄ b significat *classis a continetur in classi b*.

50. $a, b\ \varepsilon$ K . Ↄ ∴ a Ↄ b : = : $x\ \varepsilon\ a$. $Ↄ_x$. $x\ \varepsilon\ b$

[Formula b C a significare potest *classis b continet classem a*; at signo C non utiumur].

Hic signa Λ et Ↄ significationem habent quae paullo a praecedenti differt; sed nulla orietur ambiguitas. Nam si de propositionibus agatur, haec signa legantur *absurdum* et *deducitur*; si vero de classibus, *nihil* et *continetur*.

Formula $a = b$, si a et b sint classes, significat a Ↄ b . b Ↄ a. Itaque

51. $a, b\ \varepsilon$ K . Ↄ ∴ $a = b$: = : $x\ \varepsilon\ a$. $=_x$. $x\ \varepsilon\ b$

Propositiones 1...41 quoque subsistunt, si a, b... classes indicant; praeterea est:

52. $a\ \varepsilon$ b . Ↄ . $b\ \varepsilon$ K
53. $a\ \varepsilon$ b . Ↄ . $b - = \Lambda$
54. $a\ \varepsilon\ b$. $b = c$: Ↄ . $a\ \varepsilon\ c$
55. $a\ \varepsilon\ b$. b Ↄ c : Ↄ . $a\ \varepsilon\ c$

Sit s classis, et k classis quae in s contineatur; tunc dicimus k esse individuum classis s, si k ex uno tantum constat individuo. Itaque:

56. $s\ \varepsilon$ K . k Ↄ s : Ↄ :: $k\ \varepsilon\ s$. = ∴ $k - = \Lambda$: $x, y\ \varepsilon\ k$. $Ↄ_{x,y}$. $x = y$

V. De inversione.

Inversionis signum est [ ], eiusque usum in sequenti numero explicabimus. Hic tantum casus particulares exponimus.

1. Sit a propositio, indeterminatum continens x; tunc scriptura $[x]\ \varepsilon\ a$, quae legitur *ea x quibus a*, sive *solutiones*, vel *radices* conditionis a, classem significat individuis constitutam, quae conditioni a satisfaciunt. Itaque:

57. $a\ \varepsilon$ P . Ↄ : $[x\varepsilon]\ a$. ε K
58. $a\ \varepsilon$ K . Ↄ ∴ $[x\varepsilon]$. $x\ \varepsilon\ a$: = a

Notations of logic.

I. Punctuation.

By the letters $a, b, \ldots x, y, \ldots x', y', \ldots$ we indicate any indeterminate entities. Determinate entities are, however, indicated by the signs, or rather by the letters, $P, K, N, \ldots$

Generally we write signs on the same line. So that it will be clear how they are to be joined, we use *parentheses*, as in algebra, or rather *points* . : ∴ :: etc.

So that a formula divided by points may be understood, first the signs which are not seperated by points are taken together, then those seperated by one point, then those by two points, etc.

For example, let $a, b, c, \ldots$ be any signs. Then $(ab)(cd)$ means $(ab)(cd)$; and $\{[(ab)(cd)][(ef)(gh)]\}k$ means $\{[(ab)(cd)][(ef)(gh)]\}k$.

The signs of punctuation may be omitted if formulas having different punctuation have the same meaning, or if just one formula, that being the one we wish to write, has meaning.

To avoid the danger of ambiguity, we never use . : as signs of arithmetical operations.

The figure of parentheses is one ( ); if parentheses and points occur in the same formula, whatever is contained in parentheses is to be gathered first.

II. Propositions.

The sign P means *proposition*.

The sign ∧ is read *and*. Let a, b, be propositions; then $a \wedge b$ is the simultaneous affirmation of the propositions a, b. For the sake of brevity, instead of $a \wedge b$, we ordinarily write a, b.

The sign ¬ is read *not*. Let a be a P; then $\neg a$ is the negation of the proposition a.

The sign ∨ is read *or*. Let a, b be propositions; then $a \vee b$ is the same as $\neg[(\neg a) \wedge (\neg b)]$.

The sign true means *true*, or *identity*, but we never use this sign.

The sign Λ means *false*, or *absurd*.

The sign ← means *is a consequence of*. Thus $b \leftarrow a$ is read *b is a consequence of the proposition a*. But we never use this sign.

The sign → means *one deduces*; thus $a \rightarrow b$ means the same as $b \leftarrow a$. If the propositions a, b contain the indeterminate quantities $x, y, \ldots$, that is, express conditions on these objects, then $a \xrightarrow[x,y,\ldots]{} b$ means: whatever the $x, y, \ldots$, from propositions a one deduces b. If indeed there is no danger of ambiguity, instead of $\xrightarrow[x,y,\ldots]{}$, we write only →.

The sign = means *equals*. Let a, b be propositions; then $a = b$ means the same as $(a \rightarrow b) \wedge (b \rightarrow a)$; proposition $a \underset{x,y,\ldots}{=} b$ means the same as $(a \xrightarrow[x,y,\ldots]{} b) \wedge (b \xrightarrow[x,y,\ldots]{} a)$.

III. Propositions of logic.

Let $a, b, c, \ldots$ be propositions. We have:

1. $a \rightarrow a$
2. $[(a \rightarrow b) \wedge (b \rightarrow c)] \rightarrow (a \rightarrow c)$
3. $(a = b) = [(a \rightarrow b) \wedge (b \rightarrow a)]$
4. $a = a$
5. $(a = b) = (b = a)$
6. $[(a = b) \wedge (b \rightarrow c)] \rightarrow (a \rightarrow c)$
7. $[(a \rightarrow b) \wedge (b = c)] \rightarrow (a \rightarrow c)$
8. $[(a = b) \wedge (b = c)] \rightarrow (a = c)$
9. $(a = b) \rightarrow (a \rightarrow b)$
10. $(a = b) \rightarrow (b \rightarrow a)$

11. $(a \wedge b) \rightarrow a$
12. $(a \wedge b) = (b \wedge a)$
13. $(a \wedge (b \wedge c)) = ((a \wedge b) \wedge c) = (a \wedge b \wedge c)$

14. $(a \wedge a) = a$
15. $(a = b) \rightarrow [(a \wedge c) = (b \wedge c)]$
16. $(a \rightarrow b) \rightarrow [(a \wedge c) \rightarrow (b \wedge c)]$
17. $[(a \rightarrow b) \wedge (c \rightarrow d)] \rightarrow [(a \wedge c) \rightarrow (b \wedge d)]$
18. $[(a \rightarrow b) \wedge (a \rightarrow c)] = [(a \rightarrow (b \wedge c)]$
19. $[(a = b) \wedge (c = d)] \rightarrow [(a \wedge c) = (b \wedge d)]$

20. $\neg(\neg a) = a$
21. $(a = b) = [(\neg a) = (\neg b)]$
22. $(a \rightarrow b) = [(\neg b) \rightarrow (\neg a)]$

23. $(a \vee b) = \{[(\neg[(\neg a) \wedge (\neg b)]\}$
24. $[\neg(a \wedge b)] = [(\neg a) \vee (\neg b)]$
25. $[\neg(a \vee b)] = [(\neg a) \vee (\neg b)]$
26. $a \rightarrow (a \vee b)$
27. $(a \vee b) = (b \vee a)$
28. $[a \vee (b \vee c)] = [(a \vee b) \vee c] = (a \vee b \vee c)$
29. $(a \vee a) = a$
30. $[a \wedge (b \vee c)] = [(a \wedge b) \vee (a \wedge c)]$
31. $(a = b) \rightarrow [(a \vee c) = (b \vee c)]$
32. $(a \rightarrow b) \rightarrow [(a \vee c) \rightarrow (b \vee c)]$
33. $[(a \rightarrow b) \wedge (c \rightarrow d)] \rightarrow [(a \vee c) \rightarrow (b \vee d)]$
34. $[(b \rightarrow a) \wedge (c \rightarrow a)] = [(b \vee c) \rightarrow a]$

35. $[a \wedge (\neg a)] = \bot$
36. $(a \wedge \bot) = \bot$
37. $(a \vee \bot) = a$
38. $(a \rightarrow \bot) = (a = \bot)$
39. $(a \rightarrow b) = \{[(a \wedge (\neg b)] = \bot\}$
40. $\bot \rightarrow a$
41. $[(a \vee b) = \bot] = [(a = \bot) \wedge (b = \bot)]$

42. $[a \rightarrow (b \rightarrow c)] = [(a \wedge b) \rightarrow c]$
43. $[a \rightarrow (b = c)] = [(a \wedge b) = (a \wedge c)]$

Let α be the sign of some relation (eg., =, →) so that $a\ \alpha\ b$ is a proposition. Then instead of $\neg(a\ \alpha\ b)$, we write $a \not\alpha b$. Thus:

$(a \neq b) = [\neg(a = b)]$

$(a \nrightarrow b) = [(\neg(a \rightarrow b)]$

Thus the sign ≠ means *is not equal to*. If the proposition a contains the indeterminate x, $a \underset{x}{=} \bot$ means: there is an x which satisfies condition a. The sign ↛ means *one does not deduce*.

Similarly, if α and β are signs of relations, instead of $(a\ \alpha\ b) \wedge (a\ \beta\ b)$, and $(a\ \alpha\ b) \vee (a\ \beta\ b)$ we may write $a\ (\alpha \wedge \beta)\ b$ and $a\ (\alpha \vee \beta)\ b$. Thus, if a and b are propositions, the formula $a\ (\rightarrow \wedge \neq)\ b$ says: from a *one deduces* b, but not vice versa.

$[a\ (\rightarrow \wedge \neq)\ b] = [(a \rightarrow b) \wedge (b \nrightarrow a)]$

Formulas:

$[(a \rightarrow b) \wedge (b \rightarrow c) \wedge (a \nrightarrow c)] = \bot$

$[(a = b) \wedge (b = c) \wedge (a \neq c)] = \bot$

$\{(a \rightarrow b) \wedge [b\ (\rightarrow \wedge \neq)\ c]\} \rightarrow [a\ (\rightarrow \wedge \neq)\ c]$

$[a\ (\rightarrow \wedge \neq)\ b] \wedge (b \rightarrow c)\} \rightarrow [a\ (\rightarrow \wedge \neq)\ c]$

But we shall rarely use these notations.

IV. Classes.

The sign CLASS means *class*, or aggregate of entities.

The sign ∈ means *is an element of*. Thus $a \in b$ is read *a is an element of b*; *A* is a CLASS means *A is a class*; *a* is a proposition means *a is a proposition*.

Instead of $\neg(a \in b)$ we shall write $a \notin b$. The sign ∉ means *is not*; thus:

44. $(a \notin b) = [\neg(a \in b)]$

The sign $(a, b, c) \in m$ means: a, b, and c are in m; thus:

45. $[(a, b, c) \in m] = [(a \in m) \wedge (b \in m) \wedge (c \in m)]$

Let A be a class. Then $\overline{A}$ means that class made up of individuals that are not in A.

46. A is a CLASS $\rightarrow [(x \in \overline{A}) = (x \notin A)]$

Let A, B be classes. Then $A \cap B$, or $A \wedge B$, is the class composed of individuals which are at the same time in a and b; $a \cup b$ is the class composed of individuals which are in a or b.

47. A, B are CLASSES $\rightarrow \{[(x \in (a \cap b)] = [(x \in a) \wedge (x \in b)]\}$
48. A, B are CLASSES $\rightarrow \{[(x \in (a \cup b)] = [(x \in a) \vee (x \in b)]\}$

The sign ∅ indicates the class which contains no individuals. Thus:

49. A is a CLASS $\rightarrow \{(A = \emptyset) = [(x \in A) \underset{\forall x}{=} \bot]\}$

[We shall not use the sign true, which indicates the class composed of all individuals being considered].

The sign ⊂ means *is contained*. Thus $a \subset b$ means *the class a is contained in the class b*.

50. A, B are CLASSES $\rightarrow \{(A \subset B) = [(x \in A) \xrightarrow[\forall x]{} (x \in B)]\}$

[The formula $B \supset A$ could mean *the class B contains the class A*, but we shall not use the sign ⊃].

The signs (⊥/∅) and (→/⊂) have meanings here which are slightly different from the preceding, but no ambiguity will arise, for if propositions are being considered, the signs are read *absurd* (⊥) and *one deduces* (→), but if classes are being considered, they are read *empty* (∅) and *is contained* (⊂).

The formula $C_1 = C_2$, if C_1 and C_2 are classes, means $(C_1 \subset C_2) \wedge (C_2 \subset C_1)$. Thus

51. C_1, C_2 are CLASSES $\rightarrow \{(C_1 = C_2) = [(x \in C_1) \underset{\forall x}{=} (x \in C_2)]\}$

Propositions 1-41 also hold if $a, b, \ldots$ indicate classes. In addition, we have:

52. $(a \in B) \rightarrow$ B is a CLASS
53. $(a \in B) \rightarrow (B \neq \emptyset)$
54. $[(a \in B) \wedge (B = C)] \rightarrow (a \subset C)$
55. $[(a \in B) \wedge (B \subset C)] \rightarrow (a \subset C)$

Let C be a class, and C_{subset} be a class which is contained in C; then we say that C_{subset} is an individual of the class C, if C_{subset} consists of only one individual. That is:

56. [C is a CLASS ∧ $(C_{subset} \subset C)] \rightarrow \left\{(C_{subset} \in C) = \{(C_{subset} \neq \emptyset) \wedge \{[(x, y) \in C_{subset}] \xrightarrow[\forall x,y]{} (x = y)\}\}\right\}$

V. The inverse.

The sign of the inverse is ‾, and we shall explain its use in the following section. Here we give some particular examples.

1. Let a be a proposition containing the indeterminate x; then the expression $\overline{x} \in a$, which is read *those x such that a*, or *solutions*, or *roots* of the condition a, indicates the class consisting of individuals which satisfy the condition a. That is:

57. a is a proposition $\rightarrow [(\overline{x} \in a)$ is a CLASS]
58. A is a CLASS $\rightarrow \{[\overline{x} \wedge (x \in A)] = A\}$

59. $\alpha \epsilon \mathrm{P} \,.\, \text{Ɔ} \therefore x \epsilon \,.\, [x\epsilon]\, \alpha : = \alpha$

Sint α, β propositiones indeterminatum continentes x; erit:

60. $[x\epsilon]\,(\alpha\,\beta) = ([x\epsilon]\,\alpha)\,([x\epsilon]\,\beta)$

61. $[x\epsilon] - \alpha = -\,[x\epsilon]\,\alpha$

62. $[x\epsilon]\,(\alpha \cup \beta) = [x\epsilon]\,\alpha \cup [x\epsilon]\,\beta$

63. $\alpha \,\text{Ɔ}_x\, \beta \,.=.\, [x\epsilon]\,\alpha \,\text{Ɔ}\, [x\epsilon]\,\beta$

64. $\alpha =_x \beta \,.=.\, [x\epsilon]\,\alpha = [x\epsilon]\,\beta$

2. Sint x, y entia quacumque; system ex ente x et ex ente y compositum ut novum ens consideramus, et signo (x, y) indicamus; similiterque si entium numerus maior fit. Sit α propositio indeterminata continens x, y; tunc $[(x,y)\epsilon]\,\alpha$ significat classem entibus (x, y) constitutam, quae conditioni α satisfaciunt. Erit:

65. $\alpha \,\text{Ɔ}_{x,y}\, \beta \,.=.\, [(x,y)\epsilon]\,\alpha \,\text{Ɔ}\, [(x,y)\epsilon]\,\beta$

66. $[(x,y)\epsilon]\,\alpha -= \Lambda \,.=\therefore [x\epsilon]\,.\,[y\epsilon]\,\alpha -= \Lambda : -= \Lambda$

3. Sit $x\ \alpha\ y$ relatio inter indeterminata x et y (ex. g. in logica relationes $x = y$, $x -= y$, $x \,\text{Ɔ}\, y$; in arithmetica $x < y$, $x > y$, etc). Tunc signo $[\epsilon\ \alpha]\ y$ ea x indicamus, quae relationi $x\ \alpha\ y$ satisfaciunt. Commoditatis causa, loco $[\epsilon]$, signo $\ni$ utimur. Ita $\ni \alpha\ y \,.=:\, [x\ \epsilon]\,.\,x\ \alpha\ y$, et signum $\ni$ legitur *qui*, vel *quae*. Ex. gr. sit y numerus; tunc $\ni < y$ classem indicat numeris x compositam qui conditioni $x < y$ satisfaciunt, scilicet, *qui sunt minores* y, vel simpliciter *minores* y. Similiter, quum signum D significet *dividit*, vel *est divisor*, formula $\ni$D significat *qui dividunt* vel divisores. Deducitur $x \epsilon \ni \alpha\ y = x\ \alpha\ y$.

4. Sit α formula indeterminate continens x. Tunc scriptura $x'[x]\alpha$, quae legitur x' *loco* x *in* α *substituto*, formulam indicat quae obtinetur si in α, loco x, x' legimus. Deducitur $x[x]\alpha = \alpha$.

5. Sit α formula, quae indeterminata $x, y, \ldots$ continet. Tunc

   $(x', y', \ldots)\,[x, y, \ldots]\,\alpha,$

59. a is a proposition $\rightarrow [(x \in \overline{x} \in a) = a]$

Let α, β be propositions containing the indeterminate x. We will have:

60. $[\overline{x} \in (\alpha \wedge \beta)] = [(\overline{x} \in \alpha) \wedge (\overline{x} \in \beta)]$

61. $(\overline{x} \notin \alpha) = [(\neg \overline{x}) \in \alpha]$

62. $[\overline{x} \notin (\alpha \cup \beta)] = [(\overline{x} \in \alpha) \cup (\overline{x} \in \beta)]$

63. $(\alpha\,(\underset{\forall x}{\longrightarrow} / \underset{\forall x}{\subset})\,\beta) = [(\overline{x} \in \alpha)\,(\rightarrow/\subset)\,(\overline{x} \in \beta)]$

64. $(\alpha \underset{\forall x}{=} \beta) = [(\overline{x} \in \alpha) = (\overline{x} \in \beta)]$

2. Let x, y be any entities. We shall consider the system composed of the entity x and the entity y as a new entity, and indicate it by the sign (x, y); and similarly if the number of entities becomes larger. Let α be a proposition containing the indeterminate x, y; then $[(x,y)\epsilon]\,\alpha$ indicates the class of entities (x, y) which satisfy the condition α. We have:

65.

66.

3. Let $x\ \alpha\ y$ be a relation between the indeterminates x and y (eg. in logic, the relations $x = y$, $x \neq y$, $x \rightarrow y$; in arithmetic, $x < y$, $x > y$, ...). Then the sign $[\epsilon\ \alpha]\ y$ denotes the x that satisfy the relation $x\ \alpha\ y$. For the sake of convenience, we use the sign $\ni$ instead of the sign $[\epsilon]$. Thus, $\ni \alpha\ y \,.=:\, [x\ \epsilon]\,.\,x\ \alpha\ y$, and the sign $\ni$ is read *the objects that*. For example, let y be a number; then $\forall x < y$ denotes the class formed by the numbers x that satisfy the condition $x < y$, that is, *the objects that are smaller than* y, or simply *the objects smaller than* y. Similarly, if the sign D means *divides* or *is a divisor of*, the formula $\ni$D means *the objects that divide* or *the divisors*. It follows that $x \epsilon \ni \alpha\ y = x\ \alpha\ y$.

4. Let α be a formula containing the indeterminate x. Then the expression $x'[x]\alpha$, which is read x' *being substituted for* x *in* α, denotes the formula obtained if, in α, we read x' for x. It follows that $x[x]\alpha = \alpha$.

5. Let α be a formula that contains the indeterminates $x, y, \ldots$ Then

   $(x', y', \ldots)\,[x, y, \ldots]\,\alpha,$

quae legitur $x'y', \ldots$ *loco* $x, y, \ldots$ *in* α *substitutis*, formulam indicat quae obtinetur si in α loco $x, y, \ldots$, litterae $x'y', \ldots$ scribantur. Deducitur $(x, y)\,[x, y]\,\alpha = \alpha$.

which is read $x', y', \ldots$ *being substituted for* $x, y, \ldots$ in α, denotes the formula obtained if, in α, the letters $x'_1, y', \ldots$ are written for $x, y, \ldots$ It follows that $(x, y)\,[x, y]\,\alpha = \alpha$.

VI. De functionibus.

Logicae notationes quae praecedunt exprimendae cuilibet arithmeticae propositioni sufficiunt, iisdemque tantum utimur. Hic notationes alias nonnullas breviter explicamus, quae utiles fieri possunt.

Sit s quaedam classis; supponimus aequalitatem inter entia systematis s definitam, quae conditionibus satisfaciat:

$a = a$

$a = b \,.=.\, b = a$

$a = b \,.\, b = c : \text{Ɔ} \,.\, a = c$

Sit φ signum, sive signorum aggregatus, ita ut si x est ens classis s, scriptura $\varphi\ x$ novum indicet ens; supponimus quoque aequalitatem inter entia $\varphi\ x$ definitam; et si x et y sunt entia classis s, et est $x = y$, supponimus deduci posse $\varphi\ x = \varphi\ y$. Tunc signum φ dicitur esse *functionis praesignum in classi* s, et scribemus $\varphi \ni F'\ s$.

$s \epsilon K \,.\, \text{Ɔ} :: \varphi\ F'\ s \,.=\therefore x, y \epsilon s \,.\, x = y : \text{Ɔ}_{x,y} \,.\, \varphi x = \varphi y$

Verum si, cum sit x quodlibet ens classis s, scriptura $x\varphi$ novum indicet ens, et, ex, $x = y$ deducitur $x\varphi = y\varphi$, tunc dicimus φ esse *functionis postsignum in classi* s et scribemus $\varphi \epsilon s'$F.

$s \epsilon K \,.\, \text{Ɔ} :: \varphi\ s\,'F \,.=\therefore x, y \epsilon s \,.\, x = y : \text{Ɔ}_{x,y} \,.\, x\varphi = y\varphi$

Exempla. Sit a numerus; tunc $a\,+$ est functionis praesignum in numerorum classe, et $+\,a$ est functionis postsignum; quicumque enim est numerus x, formulae $a + x$ et $x + a$ novos indicant numeros, et ex $x = y$ deducitur $a + x = a + y$, et $x + a = y + a$. Itaque

$a \epsilon \mathrm{N} \,.\, \text{Ɔ} : a + \,.\, \epsilon \,.\, F'\,\mathrm{N}$

$a \epsilon \mathrm{N} \,.\, \text{Ɔ} : +\,a \,.\, \epsilon \,.\, N\,'F$

Sit φ functionis praesignum in classe s. Tunc $[\varphi]y$ classem significat iis x constitutam, quae conditioni $\varphi x = y$ satisfaciunt; scilicet:

Def. $s \epsilon \mathrm{K} \,.\, \varphi \epsilon F'\ s : \text{Ɔ} : [\varphi]y \,.=.\, [x\epsilon]\,(\varphi x = y)$

The symbols of logic introduced above suffice to express any proposition of arithmetic, and we shall only these.We explain here briefly some other symbols that may be useful.

Let S be a class; we assume that equality is defined between the objects of the system s so as to satisfy the conditions:

$a = a.$

$(a = b) = (b = a)$

$[(a = b) \wedge (b = c)] \rightarrow a = c$

Let φ be a sign or an aggregate of signs such that, if x is an object of the class S, the expression $\varphi\ x$ denotes a new object; we assume also that equality is defined between the objects $\varphi\ x$; further, if x and y are objects of the class S and if $x = y$, we assume it is possible to deduce $\varphi\ x = \varphi\ y$. Then the sign φ is said to be a *function presign in the class* S, and we write $\varphi \ni F'\ s$.

...

If, x being any object of the class S, the expression $x\varphi$ denotes a new object and $x\varphi = y\varphi$ follows from $x = y$, then we say that φ is a *function postsign in the class* S, and we write $\varphi \epsilon s'$F.

...

Examples. Let a be a number; then $a+$ is a function presign in the class of numbers, and $+a$ is a function postsign; for any number x, formulas $a + x$ and $x + a$ denote new numbers; $a + x = a + y$ and $x + a = y + a$ follow from $x = y$. Thus

...

Let φ be a function presign in the class S. Then $[\varphi]y$ denotes the class composed of x that satisfy the condition $\varphi x = y$; that is,

Def.

Classis $[\varphi]y$ vel unum vel plura, vel etiam nullum individuum continere potest. Erit:

$s \epsilon \mathrm{K} \,.\, \varphi \epsilon F'\ s : \text{Ɔ} : y = \varphi\ x \,.=.\, x \epsilon [\varphi]y$

Si vero φy uno tantum constat individuo, erit $y = \varphi x \,.=.\, x = [\varphi]y$

Sit φ functions postsignum; similiter ponimus:

$s \epsilon \mathrm{K} \,.\, \varphi \epsilon s\,'F : \text{Ɔ} \therefore y\,|\varphi| = |x\epsilon|\,(x\varphi = y).$

Signum [ ] dicitur *inversionis signum*, eiusque usus nonullos in logica iam exposuimus. Nam si α est propositio indeterminatum continens x, atque A est classis individuis x composita quae conditioni α satisfaciunt, erit $x \epsilon a \,.=\, \alpha$, tunc $a = [x\epsilon]\,\alpha$, ut in V, i.

Sit α formula indeterminate continens x, sitque φ functionis praesignum, quod litterae x praepositum, formulam α gignat; scilicet sit $\alpha = \varphi\ x$; tunc erit $\varphi = \alpha[x]$, et si x' est novum ens, erit $\varphi x' = \alpha[x]x'$, scilicet, si α est formula indeterminatum continens x, tunc $\alpha[x]x'$ significat id quod obtinetur si in α, loco x, x' ponatur.

Similiter, sit α formula indeterminate continens x, sitque φ functionis postsignum, ut $x\varphi = \alpha$; deducitur $\varphi = [x]\alpha$; tunc, si x' est novum ens, erit $x'\varphi = x'[x]\alpha$, scilicet $x'[x]\alpha$ rursum indicat id quod obtinetur si in α, loco x, x' legatur, ut in V, 4.

Alios quoque usus in logica signum [] habere potest, quos breviter esponimus, quum ipsis non utamur. Sint a et b duae classes; tunc $[a\cap]b$ sive $b[\cap a]$ classes indicat x, quae conditioni $b = a \cap x$, sive $b = x \cap a$ satisfaciunt. Si b in a non continetur, nulla classis huic conditioni satifacit; si b in a continetur, signum $b[\cap a]$ omnes indicat classes quae b continent atque in $b \cup -a$ continentur.

In Arithmetica, sint a, b numeri; tunc $[b + a]$ sive $[a+]b$ numerum indicat x, qui conditioni $b = x + a$, sive $b = a + x$ satisfacit, nempe $b - a$. Similiter erit $b[\times a] = [a \times]b = b/a$. Et in analysi hoc signum usuvenire potest; itaque

$y = sin\ x \,.=.\, x\epsilon\,[sin]\ y$ (loco $x = arc\ sin\ y$).

$dF(x) = f(x)dx \,.=.\, F\ (x)\epsilon\,[d]\ f(x)dx$ (loco $F(x) = \int f(x)dx$).

Sit rursum φ functionis praesignum in classi s, sitque k classis

The class $[\varphi]y$ may contain one or several individuals, or none at all. We have:

...

But if φy consists of just one individual, we have $y = \varphi x \,.=.\, x = [\varphi]y$

Let φ be a function postsign; we write similarly:

...

The sign [ ] is called *inversion sign*, and we have already presented some of its uses in logic. If α is a proposition containing the indeterminate x and A is a class composed of the individuals x that satisfy the condition α, we have $x \epsilon A \,.=\, \alpha$, and then $A = [x\epsilon]\,\alpha$, as in V, i.

Let α be a formula containing the indeterminate x and let φ be a function presign that yields the formula α when written before the letter x; that is, let $\alpha = \varphi x$; then we have $\varphi = \alpha[x]$, and if x' is a new object, we have $\varphi x' = \alpha[x]x'$, that is, if α is a formula containing the indeterminate x, then $\alpha[x]x'$ means what is obtained when, in α, we put x' for x.

Similarly, let α be a formula containing the indeterminate x and let φ be a function postsign, such that $x\varphi = \alpha$; it follows that $\varphi = [x]\alpha$. Then, if x' is a new object, have $x'\varphi = x'[x]\alpha$; that is, $x'[x]\alpha$ again denotes what is obtained, when, in α, read x' for x, as in V, 4..

The sign [] can have other uses in logic, which we present only briefly, since we shall not use it in these ways. Let a and b be two classes; then $[a\cap]b$ sive $b[\cap a]$, or $b[\cap a]$ denotes the classes x that satisfy the condition $b = a \cap x$, or $b = x \cap a$. If b is not contained in a, no class satisfies this condition; if b is contained in a, the sign $b[\cap a]$ denotes all classes that contain b and are contained in $b \cup -a$.

In arithmetic, let a and b be numbers; then $[b + a]$ or $[a+]b$ denotes the number x that satisfies the condition $b = x + a$, or $b = a + x$, that is $b - a$. Similarly we have $b[\times a] = [a \times]b = b/a$. This sign can even find a use in analysis; thus

... ...

... ...

Let φ again be a function presign in a class S and let C be a class

in s contenta; tunc φk classem indicat omnibus φx compositam, ubi x sunt entia classis k; scilicet

Def. $s \epsilon \mathrm{K} \,.\, k \epsilon \mathrm{K} \,.\, k \,\text{Ɔ}\, s \,.\, \varphi \epsilon F'\ s : \text{Ɔ} \,.\, \varphi\ k = [y\epsilon]\,(k \,.\, [\varphi]y : -= \Lambda)$

Sive $s \epsilon \mathrm{K} \,.\, k \epsilon \mathrm{K} \,.\, k \,\text{Ɔ}\, s \,.\, \varphi \epsilon F'\ s : \text{Ɔ} \,.\, \varphi\ k = [y\epsilon]\,([x\epsilon] : x \epsilon k \,.\, [\varphi]x = y \therefore -= \Lambda)$

Def. $s \epsilon \mathrm{K} \,.\, k \epsilon \mathrm{K} \,.\, k \,\text{Ɔ}\, s \,.\, \varphi \epsilon s\,'F : \text{Ɔ} \,.\, k\ \varphi = [y\epsilon]\,(k \,.\, y\,[\varphi] : -= \Lambda)$

Itaque, si $\varphi \epsilon F's$, tunc $\varphi\ s$ classem indicat omnibus $\varphi\ x$ constitutam, ubi x sint entia classis s. Erit:

$s \in K \,.\, \varphi \in F's \,.\, y \in \varphi s : \text{Ɔ} : \varphi[\varphi]y = y$

$s \in K \,.\, a, b \in K \,.\, a \,\text{Ɔ}\, s \,.\, b \,\text{Ɔ}\, s \,.\, \varphi \in F's : \text{Ɔ} \,.\, \varphi(a \cup b) = (\varphi a) \cup (\varphi b)$

$s \in K \,.\, \varphi \in F's : \text{Ɔ} \,.\, \varphi\Lambda = \Lambda$

$s \in K \,.\, a, b \in K \,.\, b \,\text{Ɔ}\, s \,.\, a \,\text{Ɔ}\, b \,.\, \varphi \in F's : \text{Ɔ} \,.\, \varphi a \,\text{Ɔ}\, \varphi b$

$s \in K \,.\, a, b \in K \,.\, a \,\text{Ɔ}\, s \,.\, b \,\text{Ɔ}\, s \,.\, \varphi \in F's : \text{Ɔ} \,.\, \varphi(ab) \,\text{Ɔ}\, (\varphi a)(\varphi b)$

Sit a quaedam classis; tunc $a \cap K$, sive $K \cap a$, sive Ka, classes omnes indicat formae $a \cap x$, sive $x \cap a$, xa, ubi x est classis quacumque; scilicet Ka indicat classes quae in a continentur. Formula $x \in Ka$ idem significat quod $x \in K \,.\, x \,\text{Ɔ}\, a$. Hac conventione quandoque utimur; ita KN isgnificat *numerorum classem*.

Similiter, si a est classis, $K \cup a$ indicat classes quae a continent. Sit a numerus; tunc $a + N$, sive $N + a$, *numeros* indicat *numero a maiores*; $a \times N$, sive $N \times a$, sive Na indicat *multiplices numeri* a; a^N indicat *potestas numeri* a; $N^2, N^3, \ldots$ indicat *numeros quadratos*, vel *numeros cubos*,...

Functional signorum aequalitatem, productum, potestas, ita definire licet:

Def. $s \in K \,.\, \varphi, \psi \in F's : \text{Ɔ} \therefore \varphi = \psi : =: x \in s \,.\, \text{Ɔ} \,.\, \varphi x = \psi x$

Def. $s \in K \,.\, \varphi \in F's \,.\, \psi \in F'\varphi s \,.\, x \in s : \text{Ɔ} \,.\, \psi\varphi x = \psi(\varphi x)$

Itaque, in definitionis hypothesi, erit $\psi\varphi$ novum functionis praesignum; idque *productum signorum* ψ *et* φ vocatur.

Similiterque, si φ, ψ sunt functionis postsigna.

Haec valet propositio:

$s \in K \,.\, \varphi \in F's \,.\, \varphi s \,\text{Ɔ}\, s : \text{Ɔ} : \varphi\varphi s \,\text{Ɔ}\, s \,.\, \varphi\varphi\varphi s \,\text{Ɔ}\, s.$ etc.

Funcitones $\varphi\varphi, \varphi\varphi\varphi, \ldots$ *iteraiae* vocantur, et communiter signis $\varphi^2, \varphi^3, \ldots$ indicantur, ut operationis φ potestates.

contained in S; then φC denotes the class consisting of all φx, where the x are the objects of class C; that is

Def.

Or

Def.

Thus, if $\varphi \epsilon F'S$, then $\varphi\ s$ denotes the class composed of all $\varphi\ x$, where the x are objects of the class S. We have:

...

...

...

...

...

Let a be a class; then $a \cap K$, or $K \cap a$, or Ka, denotes all classes of the form $a \cap x$, or $x \cap a$, xa, where x is any class; that is Ka denotes the classes that are contained in a. The formula $x \in Ka$ means the same as $x \in K \,.\, x \,\text{Ɔ}\, a$. We shall sometimes use this convention; thus KN means *a class of numbers*.

Similarly, if A is a class, $K \cup A$ indicates the classes that contain A. Let a be a number; then $a + N$, or $N + a$, denotes *the numbers greater than the number* a; $a \times N$, or $N \times a$, or Na denotes *the multiples of the number* a; a^N denotes *the powers of the number* a; $N^2, N^3, \ldots$ denote *the squares*, *the cubes*,...

Equality, product and powers can be defined thus for function signs:

Def.

Def.

Thus, if we assume this definition, we have the new function presign $\psi\varphi$; it is called the *product of the signs* ψ and φ.

Similarly if φ, ψ are function postsigns.

The following proposition holds:

...

The functions $\varphi\varphi, \varphi\varphi\varphi, \ldots$ are said to be *iterated* and are generally denoted by the signs $\varphi^2, \varphi^3, \ldots$ as powers of the operation φ.

Si vero φ est functionis postsignum, ha faciliori notatione, absque ambiguitate, uti licet:

Def. $s \in K \,.\, \varphi \in s'F \,.\, s\varphi \,\text{Ɔ}\, s : \text{Ɔ} : \varphi 1 = \varphi \,.\, \varphi 2 = \varphi\varphi \,.\, \varphi 3 = \varphi\varphi\varphi.$ etc.

In definitionis hypothesi, si $m, n \in N$, erit $\varphi\ (m + n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

Si hac definitione in Arithmetica utimur, haec invenimus. Numerum qui sequitur numerum a signo faciliori $a+$ indicare possumus; tunc $a + 1, a + 2, \ldots$ et, si b est numerus, $a + b$, sensum habent $a+, a++, \ldots$ quod a definitione in §1 patet. Propositionem 6 in §1 scribere possumus $N + \,\text{Ɔ}\, N$. Si a, b, c sunt numeri, tunc $a : +b \,.\, c$ significat $a + bc$, et $a : \times b \,.\, c$ significat ab^c.

Multi aliis proprietatibus gaudent functionem signa, praesertim si conditioni satisfaciunt: $\varphi x = \varphi y \,.\, \text{Ɔ} \,.\, x = y$. Functionis signum quod huic conditioni satisfacit vocatur a clarrissimo Dedekind *simile* (ähnliche Abbildung).

Sed his exponendis locus deest.

But if φ is a function postsign, we can use the following more convenient notation without ambiguity:

Def.

Assuming this definition, if $m, n \in N$, we have $\varphi\ (m + n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

If we use this definition in arithmetice, we obtain the following. We can denote the number that follows the number a by the more convenient sign $a+$; then $a + 1$, $a + 2, \ldots$, and, if b is a number, $a + b$, have the meaning of $a+$, $a++, \ldots$, which is clear from the definition in §1 below. Proposition 6 in §1 can be written $N + \,\text{Ɔ}\, N$. If a, b, c are numbers, then $a : +b \,.\, c$ means $a + bc$, and $a : \times b \,.\, c$ means ab^c.

Function signs possess many other properties, especially if they satisfy the condition: $\varphi x = \varphi y \,.\, \text{Ɔ} \,.\, x = y$. A function sign that satisfies this condition is called *equivalent*[8] by Dedekind (ähnliche Abbildung).

But we lack the space to present these properties.

Declarationes.

Defenitio, vel breviter *Def.* est propositio formam habens $x = a$, sive $\alpha \,\text{Ɔ}\,.\, x = a$, ubi α est signorum aggregatus sensum habens notum; x est signum, vel signorum aggregatus significatione adhuc carnes; α vero est conditio sub qua definitio datur.

Theorema, (Theor. vel Th) est propositio quae demonstratur. Si theorema formam habet $\alpha \,\text{Ɔ}\, \beta$, ubi α et β sunt propositiones, tunc α dicitur *Hypothesis* (Hyp. vel breviter Hp.), β vero *Thesis* (Thes. vel Ts.). Hyp. ac Ts. a Theorematis forma pendent; nam si loco $\alpha \,\text{Ɔ}\, \beta$ scribemus $-\beta \,\text{Ɔ}\, -\alpha$, erit $-\beta$ Hp, et $-\alpha$ Ts.; si vero scribemus $\alpha - \beta = \Lambda$, Hp. ac Ts. absunt.

In quolibet § signum P quod quidam numerus sequatur, propositionem indicat eiusdem § hoc numero signatam. Logicae propositiones indicantur signo L et propositiones numero.

Formulae quae in una linea non continentur, in altera linea, nullo interposito signo, sequuntur.

Remarks.

A *Definition*, or *Def.* for short, is a proposition of the form $x = a$ or $\alpha \,\text{Ɔ}\,.\, x = a$, where a is an aggregate of signs having a known meaning, x is a sign or an aggregate of signs, hitherto without meaning, and α is the condition under which the definition is given.

A *theorem* (Theor. or Th.) is aproposition that is *proved*. If a theorem has the form $\alpha \,\text{Ɔ}\, \beta$, where α and β are propositions, then α is called the *hypothesis* (Hyp., or even shorter, Hp.) and β the *thesis* (Thes. or Ts.). Hyp. and Ts. depend on the form of the theorem; in fact, if we write $\alpha \,\text{Ɔ}\, \beta$ instead of $-\beta \,\text{Ɔ}\, -\alpha$, then $-\beta$ is the Hp, abd $-\alpha$ the Ts.; if we write $\alpha - \beta = \Lambda$, Hp. and Ts. do not exist.

In any § below, the sign P followed by a number denotes the proposition indicated by that number in the same §. Propositions of logic are indicated by the sign L and the number of the proposition.

Formulas that do not fit on one line are continued on the next line without any intervening sign.

Arithmetices Principia.

§1. De numeris et de additione.

Explicationes.

Signo	N	significatur	*numerus (integer positivus).*
»	1	»	*unitas.*
»	$a + 1$	»	*sequens* a, sive a *plus* 1.
»	=	»	*est aequalis*. Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.

Axiomata.

1. $1 \in N$
2. $a \in \mathrm{N} \,.\, \text{Ɔ} \,.\, a = a$
3. $a, b, c \in \mathrm{N} \,.\, \text{Ɔ} : a = b \,.=.\, b = a$
4. $a, b \in \mathrm{N} \,.\, \text{Ɔ} \therefore a = b \,.\, b = c : \text{Ɔ} \,.\, a = c$
5. $a = b \,.\, b \in \mathrm{N} : \text{Ɔ} \,.\, a \in N$
6. $a \in \mathrm{N} \,.\, \text{Ɔ} \,.\, a + 1 \in N$
7. $a, b \in \mathrm{N} \,.\, \text{Ɔ} : a = b \,.=.\, a + 1 = b + 1$
8. $a \in \mathrm{N} \,.\, \text{Ɔ} \,.\, a + 1 -= 1$
9. $k \in K \therefore 1 \in k \therefore x \in \mathrm{N} \,.\, x \in k : \text{Ɔ}_x \,.\, x + 1 \in k :: \text{Ɔ} \,.\, \mathrm{N} \,\text{Ɔ}\, k$

Definitiones.

The Principles of Arithmetic.

§1. Numbers and addition.

Explanations.

The sign	N	means	*number (positive integer).*
»	1	»	*unity.*
»	$a + 1$	»	*the successor of* a, or a *plus* 1.
»	=	»	*is equal to*. This must be considered as a new sign, although it has the appearance of a sign of logic.

Axioms.

...

...

...

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...

Definitions.

10. $2=1+1;\ 3=2+1;\ 4=3+1;$ etc. ...

2

Theoremata.

Theorems.

11. $2 \in N.$ ...

Demonstratio:

Proof:

$P1.\supset:$	$1 \in N$	(1)	....	...	(1)
$1[a](P6).\supset:$	$1 \in \mathrm{N}.\supset.1+1 \in N$	(2)	...	...	(2)
$(1)(2).\supset:$	$1+1 \in N$	(3)	...	...	(3)
$P10.\supset:$	$2=1+1$	(4)	...	...	(4)
$(4).(3).(2,1+1)[a,b](P5):\supset:$	$2 \in N$	(Theor.)			

Nota. - Huius facillimae demonstrationis gradus omnes ecplicite scripsimus. Brevitatis causa ipsam ita scribemus:

Note. - We have explicitly written every step of this very easy proof. For the sake of brevity, we shall write it as follows:

$P1.1[a](P6):\supset:1+1 \in \mathrm{N}.P10.(2,1+1)[a,b](P5):\supset:Th$ ...

vel ...

$P1.P6:\supset:1+1 \in \mathrm{N}.P10.P5:\supset:Th$ ...

12. $3,4,\ldots \in N$ ...

13. $a,b,c,d \in \mathrm{N}.a=b.b=c.c=d:\supset:a=d$ ...

Dem. $Hyp.P4:\supset:a,c,d \in \mathrm{N}.a=c.c=d.P4:\supset:Thes.$ ...

14. $a,b,c \in \mathrm{N}.a=b.b=c.a -= c:=\Lambda$ ...

Dem. $P4.L39:\supset.Theor.$ ...

15. $a,b,c \in \mathrm{N}.a=b.b -= c:\supset.a -= c$ ...

16. $a,b \in \mathrm{N}.a+1=b+1:\supset.a=b$ ...

17. $a,b \in \mathrm{N}.\supset:a -= b.=.a+1 -= b+1$ ...

Dem. $P7.L21:\supset.Theor$ ...

Definitio.

Definition.

18. $a,b \in \mathrm{N}.\supset.a+(b+1)=(a+b)+1$ ...

Nota. - Hanc definitionem ita legere oportet: si a et b sunt numeri, et $(a+b)+1$ sensum habet (scilicet si $a+b$ est numerus), sed $a+(b+1)$ nondum definitus est, tunc $a+(b+1)$ significat numerum qui $a+b$ sequitur.

Note. - This definition should be read: if a and b are numbers, and $(a+b)+1$ has meaning (that is, if $a+b$ is a number), but $a+(b+1)$ has not yet been defined, then $a+(b+1)$ indicates the number that follows $a+b$.

Ab hac definitione, et a praecedentibus deducitur:

From this definition, and the preceding, we deduce that:

$a \in \mathrm{N}.\supset\therefore a+2=a+(1+1)=(a+1)+1$ ...

$a \in \mathrm{N}.\supset\therefore a+3=a+(2+1)=(a+2)+1,$ etc.

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Theoremata.

Theorems.

19. $a,b \in \mathrm{N}.\supset.a+b \in N.$ ... ...

Dem. $a \in \mathrm{N}$. P6 $:\supset:a+1 \in \mathrm{N}:\supset:1 \in [b\,\in]$ Ts. (1) ... ...

$a \in \mathrm{N}.\supset::b \in \mathrm{N}.b \in [b\,\in]$ Ts $:\supset:a+b \in \mathrm{N}$. P6 $:\supset:(a+b)+1 \in \mathrm{N}$. (2) ... ...

P18 $:\supset:a+(b+1) \in \mathrm{N}:\supset:(b+1) \in [b\,\in]$ Ts.

$a \in \mathrm{N}.(1).(2).\supset::1 \in [b\,\in]$ Ts $\therefore b \in \mathrm{N}.b \in [b\,\in]$ Ts $:\supset:b+1 \in [b\,\in]$ (3) ... ...

Ts $\therefore([b\,\in]$ Ts $)[k]$ P9 $::\supset:\mathrm{N}\supset[b\,\in]$ Ts . (L50) $::\supset:b \in \mathrm{N}.C$ Ts.

(3). (L42) $:C:a,b \in \mathrm{N}.\supset.$ Thesis (Th.) ... ...

20. *Def.* $a+b+c=(a+b)+c$ ... ...

21. $a,b,c \in \mathrm{N}.\supset.a+b+c \in N$ ... ...

22. $a,b,c \in \mathrm{N}.\supset:a=b.=.a+c=b+c$ ... ...

Dem. $a,b \in \mathrm{N}$. P7 $:\supset.1 \in [c\,\in]$ Ts. (1) ... ...

$a,b \in \mathrm{N}.\supset::c \in \mathrm{N}.c \in [c\,\in]$ Ts (2) ... ...

$\therefore\supset\therefore a=b.=.a+c=b+c:a+c,b+c \in \mathrm{N}:a+c=b+c.=.a+c+$ $1=b+c+1\therefore\supset\therefore a=b.=.a+(c+1)=b+(c+1)\therefore\supset\therefore(c+1) \in [C\,\in]$ Ts.

$a,b \in \mathrm{N}.(1).(2):\supset::1 \in [c\,\in]$ Ts $\therefore c \in [c\,\in]$ Ts $.\supset.(c+1) \in [c\,\in]$ Ts (3) ... ...

$::\supset::c \in \mathrm{N}.\supset.$ Ts.

23. $a,b,c \in \mathrm{N}.\supset.a+(b+c)=a+b+c$ ... ...

Dem. $a,b \in \mathrm{N}$. P18 . P20 $:\supset.1 \in [c\,\in]$ Ts. (1) ... ...

$a,b \in \mathrm{N}.\supset\therefore c \in \mathrm{N}.c \in [c\,\in]$ Ts $:\supset:a+(b+c)=a+b+c$. P7 (2) ... ...

$:\supset:a+(b+c)+1=a+b+c+1$. P18

$:\supset:a+(b+(c+1))=a+b+(c+1):\supset.c+1 \in [c\,\in]$ Ts.

(1)(2) (P9) $.\supset.$ Theor. ... ...

24. $a \in \mathrm{N}.\supset.1+a=a+1$ ... ...

Dem. P2 $.\supset.1 \in [a\,\in]$ Ts. (1) ... ...

$a \in \mathrm{N}.a \in [a\,\in]$ Ts (2) ... ...

$:\supset:1+1=a+1:\supset:1+(a+1)=(a+1)+1:\supset:(a+1) \in [a\,\in]$ Ts.

(1)(2)$.\supset.$ Theor. ... ...

24'. $a,b \in \mathrm{N}.\supset.1+a+b=a+1+b$ ... ...

Dem. Hyp. P24 $:\supset\supset 1+a=a+1$. P22 $:\supset.$ Thesis. ... ...

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25. $a,b\breve{a} \in \mathrm{N}.\supset.a+b=b+a.$ ... ...

Dem. $a \in \mathrm{N}$. P24 $:\supset:1 \in [b\,\in]$ Ts. (1) ... ...

$a \in \mathrm{N}.\supset\therefore b \in \mathrm{N}.b \in [b\,\in]$ Ts $:\supset:a+b=b+a$. P7 (2) ... ...

$:\supset:(a+b)+1=(b+a)+1.(a+b)+1=(a+(b+1).(b+a)+1=$ $1+(b+a).1+(b+a)=(1+b)+a.(1+b)+a=(b+1)+a:\supset:$ $a+(b+1)=(b+1)+a:\supset:(b+1) \in [b\,\in]$ Ts.

(1)(2)$.\supset.$ Theor. ... ...

26. $a,b,c \in \mathrm{N}.\supset:a=b.=.c+a=c+b.$ ... ...

27. $a,b,c \in \mathrm{N}.\supset:a+b+c=a+c+b.$ ... ...

28. $a,b,c,d \in \mathrm{N}.a=b.c=d:\supset.a+c=b+d.$ ... ...

§2. De substractione.

§2. Substraction.

Explicationes.

Explanations.

Signo	$-$	legitur	*minus.*	The sign	$-$	is read	*minus.*
»	$<$	»	*est minor.*	»	$<$	»	*is less than.*
»	$>$	»	*est maior.*	»	$>$	»	*is greater than.*

Definitiones.

Definitions.

1. $a,b \in \mathrm{N}.\supset:b-a=\mathrm{N}[x\,\in](x+a=b).$

2. $a,b \in \mathrm{N}.\supset:a<b.=.b-a -= \Lambda.$

3. $a,b \in \mathrm{N}.\supset:b>a.=.a<b.$

$a+b-c=(a+b)-c;a-b+c=(a-b)+c;a-b-c=(a-b)-c.$

Theoremata.

Theorems.

4. $a,b,a',b' \in \mathrm{N}.a=a'.b=b':\supset:b-a=b'-a'.$

Dem. Hyp $.\supset:x+a=b.=.x+a'=b':\supset.$ Thesis.

5. $a,b \in \mathrm{N}.\supset:a<b.=.b-a \in N.$

Dem. $a,b \in \mathrm{N}:\supset\therefore x,y \in b-a.\supset_{x,y}:x,y \in \mathrm{N}.x+a=b.y+a=b.$ §1 P22 (1)

$:\supset:x=y.$

$a,b \in \mathrm{N}.a<b.$ P2 (2)

$.(1):\supset\therefore b-a -= \Lambda:x,y \in b-a.\supset.x=y:(N,b-a)[s,k]$ (L56)

∴Ↄ∴ b−a∈N.

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a,b∈N . b−a∈N . (L56) :Ↄ: b−a −= Λ :Ↄ: a<b. (3)

(2)(3).Ↄ. Theor.

6. a,b∈N . a<b :Ↄ. b−a+a=b.

Dem. Hyp . P5 . P1 :Ↄ: b−a∈N . (b−a)∈[x∈](x+a=b) :Ↄ: Thes.

7. a,b,c∈N .Ↄ: c=b−a .=. c+a=b.

Dem. Hyp .§1 P22 . P6 :Ↄ: c=b−a .=. c+a=b−a+a .=. c+a=b.

8. a,b∈N .Ↄ. a+b−a=b.

Dem. (a+b,b)[b,c] P7 .Ↄ. Theor.

9. a,b,c∈N . a<b :Ↄ: c+(b−a)=c+b−a.

Dem. Hyp . P6 :Ↄ: (b−a)+a=b :Ↄ: c+(b−a)+a=c+b . P7 :Ↄ: Thesis.

10. a,b,c∈N . a>b+c :Ↄ. a−(b+c)=a−b−c.

11. a,b,c∈N . b>c . a>b−c :Ↄ. a−(b−c)=a+c−b.

12. a,b,a′,b′∈N . a=a′ . b=b′ :Ↄ: a<b .=. a′<b′.

Dem. Hyp .Ↄ. b−a=b′−a′ .Ↄ. b−a∈N = b′−a′∈N .Ↄ. Thes.

13. a,b∈N .Ↄ. a<a+b.

Dem. Hyp . P8 :Ↄ: a+b−a=b :Ↄ. a+b−a∈N . P5 :Ↄ: Thesis.

14. a,b,c∈N . a<b . b<c :Ↄ. a<c.

Dem. Hyp .Ↄ: b−a∈N . c−b∈ :Ↄ: (b−a)+(c−b)∈N :Ↄ: c−a∈N :Ↄ. Thesis.

15. a,b,c∈N .Ↄ: a<b .=. a+c<b+c.

Dem. Hyp .Ↄ: a<b .=. b−a∈N .=. (b+c)−(a+c)∈N .=. a+c<b+c.

16. a,b,a′,b′∈N . a<b . a′<b′ :Ↄ. a+a′<b+b′.

Dem. Hyp .Ↄ: a+a′<b+a′ . b+a′<b+b′ :Ↄ. Thesis.

17. a,b,c∈N . a<b<c :Ↄ. c−a>c−b.

Dem. Hyp .Ↄ b−a∈N . c−b∈N . (c−b)+(b−a)=c−a :Ↄ. Thesis.

18. a∈N .Ↄ: a=1 .∪. a>1.

Dem. 1∈[a∈] Thesis.

a∈N . P13 :Ↄ: a+1>1 :Ↄ: a+1∈[a∈] Thesis.

(1)(2).Ↄ. Theor.

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19. a,b∈N .Ↄ. a+b −= b.ă

Dem. a∈N .§1 P8 :Ↄ: a+1 −= 1 :Ↄ: 1∈[b∈] Thesis. (1)

a∈N . b∈N . b∈[b∈] Ts :Ↄ: a+b −= b .§1 P17 (2)

:Ↄ: a+(b+1) −= b+1 :Ↄ: b+1∈[b∈] Ts.

(1)(2).Ↄ. Theor.

20. a,b∈N . a<b . a=b := Λ.

Dem. Hyp :Ↄ: b−a∈N . (b−a)+a=a . P19 :Ↄ: Λ.

21. a,b∈N . a>b . a=b := Λ.

22. a,b∈N . a>b . a<b := Λ.

23. a,b∈N :Ↄ: a<b .∪. a=b .∪. a>b.

Dem a∈N . P18 :Ↄ. 1∈[b∈] Ts. (1)

a,b∈N . a<b :Ↄ. a<b+1. (2)

a,b∈N . a=b :Ↄ. a<b+1. (3)

a,b∈N . a>b :Ↄ: a−b∈N . P18 :Ↄ: a−b=1 .∪. a−b>1. (4)

a,b∈N . a−b=1 :Ↄ. a=b+1. (5)

a,b∈N . a−b>1 :Ↄ. a>b+1. (6)

a,b∈N . a>b .(4)(5)(6) :Ↄ: a=b+1 .∪. a>b+1. (7)

a,b∈N : a>b .∪. a=b .∪. a>b : (2)(3)(7) ∴Ↄ∴ a<b+1 .∪. a = b+1 .∪. a>b+1. (8)

a,b∈N . b∈[b∈] Ts .(8) :Ↄ: b+1∈[b∈] Ts. (9)

(1)(9).Ↄ. Theor.

§3. De maximis et minimis. | §3. Maxima and minima.

Explicationes. | Explanations.

Sit a∈KN, hoc est sit *a* quaedam numerorum classis; tunc *Ma* legatur *maximus inter a*, et *Wa* legatur *minimus inter a*.

Let a∈K N, that is, let *a* be a class of numbers; then M*a* is read *greatest among a*, and W*a* is read *least among a*.

Definitiones. | Definitions.

1. a∈K N .Ↄ: Ma=[x∈](x∈a ∴ a.∈>x := Λ.ă

2. a∈K N .Ↄ: Wa=[x∈](x∈a ∴ a.∈<x := Λ.ă

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Theoremata. | Theorems.

3. n∈N . a∈K N . a −= Λ . a϶>n = Λ :Ↄ. Ma∈N.ă

Dem. a∈K N . a −= Λ . a϶>1 = Λ :Ↄ: a=1 :Ↄ. Ma=1 :Ↄ. Ma∈N.ă (1)

(1)Ↄ: 1∈[n∈] (Hp Ↄ Ts). (2)

n∈N . a∈K N . a϶>n+1 = Λ . n+1∈a :Ↄ: n+1 = Ma :Ↄ: Ma∈N. (3)

n∈N . a∈K N . a϶>n+1 = Λ . n+1 −∈ a :Ↄ: a϶>n = Λ. (4)

n∈[n∈] (Hp Ↄ Ts) . a∈K N . a϶>n+1 = Λ . n+1 −∈ a :Ↄ: Ma∈N. (5)

n∈[n∈] (Hp Ↄ Ts) .(6) :Ↄ. (n+1)∈[n∈] (Hp Ↄ Ts). (7)

(2)(7) .§1 P9 :Ↄ: n∈N .Ↄ. Hp Ↄ Ts. (Th.)

4. a∈K N . a −= Λ :Ↄ. Wa∈N.

5. a∈K N .Ↄ: Wa = M[x∈](a϶<x = Λ).

§4. De multiplicatione. | §4. Multiplication.

Definitiones. | Definitions.

1. a∈N .Ↄ. a×1=a.ă

2. a,b∈N .Ↄ. a×(b+1)=a×b+a.

ab=a×b; ab+c=(ab)+c; abc=(ab)c. ...

Theoremata. | Theorems.

3. a,b∈N .Ↄ. ab∈N.ă

Dem. a,b∈N . P1 :Ↄ: a×1∈N :Ↄ. 1∈[b∈] Ts.

a,b∈N . b∈[b∈] Ts :Ↄ: a×b∈N .§1 P19 :Ↄ: ab+a∈N . P1 (2)

:Ↄ: a(b+1)∈N :Ↄ: b+1∈[b∈] Ts.

(1)(2).Ↄ. Theor.

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4. a,b,c∈N .Ↄ. (a+b)c = acśbc.ă

Nota. Haec est prop. 5^a Euclidis elem. libri VII. | *Note.* This is prop. 5^a of Euclid's elem. book VII.

Dem. a,b∈N . P1 :Ↄ: 1∈[c∈] Ts. (1)

a,b,c∈N . c∈[c∈] Ts :Ↄ: (a+b)c=ac+bc .§1 P22 (2)

:Ↄ: (a+b)c+a+b = ac+bc+a+b . P2

:Ↄ: (a+b)(c+1)=a(c+1)+b(c+1) :Ↄ: c+1∈[c∈] Ts.

(1)(2).Ↄ. Theor.

5. a∈N .Ↄ. 1×a=a.

Dem. 1∈[a∈] Ts. (1)

a∈[a∈] Ts .Ↄ. 1×a=a .Ↄ. 1×a+1 = (2)

a+1 .×. 1×(a+1)=a+1 .Ↄ. a+1∈[a∈] Ts.

(1)(2).Ↄ. Theor.

6. a,b∈N .Ↄ. ba+a=(b+1)a.

7. a,b∈N .Ↄ. ab=ba. (Eucl. VII, 16)

Dem. a∈N . P5 . P1 :Ↄ. a×1=a=1×a :Ↄ: 1∈[b∈] Ts. (1)

a,b∈N . b∈[b∈] Ts (2)

:Ↄ: ab=ba :Ↄ: ab+a=ba+a . P1

.P6 :Ↄ: a(b+1)=(b+1)a :Ↄ: b+1∈[b∈] Ts.

(1)(2).Ↄ. Theor.

8. a,b,c∈N .Ↄ. a(b+c)=ab+ac.

Dem. P4 . P7 :Ↄ. Theor.

9. a,b,c∈N . a=b :Ↄ: ac=bc.

Dem. a,b∈N . a=b ::Ↄ:: 1∈[c∈] Ts ∴ c∈[c∈] Ts .Ↄ: ac=bc . a=b :Ↄ: ac+a=bc+b :Ↄ: a(c+1)=b(c+1) :Ↄ: c+1∈[c∈] Ts ::Ↄ: c∈N .Ↄ. Ts.

10. a,b,c∈N : a<b :Ↄ. (b−a)c=bc−ac. (Eucl. VII, 7)

Dem. Hyp .Ↄ: b−a∈N . (b−a)+a=b :Ↄ: (b−a)c+ac = bc :Ↄ: (b−a)c=bc−ac.

11. a,b,c∈N . a<b :Ↄ: ac<bc.

Dem. Hyp .Ↄ: b−a∈N . P3 :Ↄ: (b−a)c∈N . P10 :Ↄ: bc−ac∈N :Ↄ Thesis.

12. a,b,c∈N .Ↄ ∴ a<b .=. ac<bc : a=b .=. ac=bc : a>b .=. ac>bc.

13. a,b,a′,b′∈N . a<a′ . b<b′ :Ↄ: ab<a′b′.

14. a,b∈N :Ↄ. ab .>∪=. a.

15. a,b,c∈N .Ↄ. a(bc)=abc.

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Dem. a,b∈N . P1 :Ↄ: 1∈[c∈] Ts. (1)

a,b,c∈N . c∈[c∈] Ts :Ↄ: a(bc)=abc :Ↄ: (2)

a(bc)+ab=abc+ab :Ↄ: a(bc+b)=ab(c+1) :Ↄ: a(b(c+1))=ab(c+1) :Ↄ: c+1∈[c∈] Ts.

(1)(2).Ↄ. Theor.

§5. De potestatibus. | §5. Powers.

Definitiones. | Definitions.

1. a∈N .Ↄ. $a^1=a$.

2. a,b∈N .Ↄ. $a^{b+1}=a^b a$.

Theoremata. | Theorems.

3. a,b∈N .Ↄ. a^b∈N.

Dem. a∈N . P1 :Ↄ. 1∈[b∈] Ts. (1)

a,b∈N . b∈[b∈] Ts :Ↄ: a^b∈N .§4 P3 :Ↄ: $a^b a$∈N . (2)

P1 :Ↄ: a^{b+1}∈N :Ↄ: b+1∈[b∈] Ts.

(1)(2).Ↄ. Theor.

4. a∈N .Ↄ. $1^a=1$.

5. a,b,c∈N .Ↄ. $a^{b+c}=a^b a^c$.

6. a,b,c∈N .Ↄ. $(ab)^c=a^c b^c$.

7. a,b,c∈N .Ↄ. $(a^b)^c=a^{bc}$.

8. a,b,c∈N .Ↄ ∴ a<b .=. $a^c<b^c$: a=b .=. $a^c=b^c$: a>b .=. $a^c>b^c$.

9. a,b,c∈N . a>1 .Ↄ ∴ b<c .=. $a^b<a^c$: b=c .=. $a^b=a^c$: b>c .=. $a^b>a^c$.

§6. De divisione. | §6. Division.

Explicationes. | Explanation.

Signum		legatur	The sign		is read
/		divisus per	/		divided by
D	=	divisibi, sive est divisor	D	=	divides, or is a divisor of
ꓷ	=	est multiplex	ꓷ	=	is a multiple of
Np	=	numerus primus	Np	=	prime number
π	=	est primus cum	π	=	is prime with

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Definitiones. | Definitions.

1. a,b∈N .Ↄ: b/a = N[x∈](xa=b).

2. a,b∈N .Ↄ: aDb .=. b/a −= Λ.

3. a,b∈N .Ↄ: bꓷa .=. aDb.

4. Np = N[x∈](϶Dx . ϶>1 . ϶<x := Λ).

5. a,b∈N .Ↄ:: aπb ∴=∴ ϶Da . ϶Db . ϶>1 := Λ.

6. a,b∈N .Ↄ ∴ ϶D(a,b) :=: ϶Da .∩. ϶Db.

7. a,b∈N .Ↄ ∴ ϶ꓷ(a,b) :=: ϶ꓷa .∩. ϶ꓷb.

ab/c=(ab)/c; a/b/c=(a/b)/c; a/b×c=(a/b)c.

Theoremata. | Theorems.

Nota. Haec theoremata ut in substractione demonstrantur. | *Note.* These theorems are proved as for subtraction.

8. a,b,a′,b′∈N . a=a′ . b=b′ :Ↄ. a/b=a′/b′.

9. a,b,a′,b′∈N . a=a′ . b=b′ :Ↄ: aDb .=. a′Db′.

10. a,b,c∈N .Ↄ: ac=b .=. c=b/a.

11. a,b∈N .Ↄ: aDb .=. b/a∈N.

12. a∈N .Ↄ. a/1=a.

13. a∈N .Ↄ. a/a=1.

14. a∈N .Ↄ. 1Da.

15. a∈N .Ↄ. aDa.

16. a,b∈N . ab/b=a.

17. a,b∈N . aDb :Ↄ. a(b/a)=b.

18. a,b,c∈N . cDb :Ↄ. a(b/c)=ab/c.

19. a,b,c∈N . aꓷbc :Ↄ: a/(bc)=a/b/c.

20. a,b,c∈N . aꓷb . bꓷc :Ↄ. a/(b/c)=a/b×c.

21. a,m,n∈N . m>n :Ↄ. $a^m/a^n=a^{m-n}$.

22. a,b∈N .Ↄ. aDab.

23. a,b,c∈N . aDb . bDc :Ↄ. aDc.

24. a,b,c∈N . aDbDc :Ↄ. c/aꓷc/b.

25. a,b,c∈N . cDa . cDb :Ↄ. (a+b)/c=a/c+b/c.

26. a,b,c∈N . cDa . cDb . a>b :Ↄ: (a−b)/c=a/c−b/c.

27. a,b,c,∈N . cDa . cDb :Ↄ. cDa+b.

28. a,b,c∈N . cDa . cDb . a>b :Ↄ. cDa−b.

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29. a,b,c,m,n∈N . cDa . cDb :Ↄ. cDma+nb.

30. a,b,c,m,n∈N . cDa . cDb . ma>nb : Ↄ. cDma−nb.

31. a,b∈N . aDb :Ↄ: a .<∪=. b.

Dem. Hyp . P11 . P17 . §4 P14 :Ↄ: b/a∈N . a(b/a)=b . a<∪=a(b/a) :Ↄ. Thesis.

32. a,b∈N . aDb . bDa :Ↄ. a=b.

33. a∈N .Ↄ. M϶Da=a.

34. a,b∈N . a>b :Ↄ. ϶D(a,b) = ϶D(b,a−b).

Dem. Hyp. P28 :Ↄ ∴ xDa . xDb :Ↄ: xDb . xD(a−b) (1)

Hyp. P27 :Ↄ ∴ xDb . xD(a−b) :Ↄ: xDb . xD(b+(a−b)) :Ↄ: xDb . xDa. (2)

(1)(2)Ↄ: Hyp. Ↄ ∴ xDa . xDb :=: xDb . xD(a−b). (Th.)

35. a,b∈N .Ↄ: M϶D(a,b)∈N.

Dem. 1Da . 1Db :Ↄ: ϶D(a,b) −= Λ. (1)

϶D(a,b) . ϶>a := Λ. (2)

(1)(2) .§3 P3 :Ↄ. Th.

36. a,b∈N .Ↄ. ϶D(a,b) = ϶DM϶D(a,b). (Eucl. VII, 2)

Dem. k = N[c∈] (Hp. a<c . b<c :Ↄ. Ts.). (1)

a∈N . b∈N . a<1 . b<1 := Λ. (2)

(1)(2).Ↄ. 1∈K. (3)

a,b∈N . a<c+1 . b<c+1 :Ↄ ∴ a<c . b<c :∪: a = c . b<c :∪: a<c . b=c :∪: a=c . b=c. (4)

c∈k . a,b∈N . a<c . b<c :Ↄ: Ts. (5)

c∈k . a,b∈N . a=c . b<c :Ↄ: c∈ k . b<c/pa−b<c . ϶D(a,b) = ϶D(b,a−b) :Ↄ: ϶D(b,a−b) = ϶Dm϶D(b,a−b) :Ↄ: ϶D(a,b) = ϶DM϶D(a,b) :Ↄ: Ts. (6)

(a,b)[b,a](6)Ↄ. c∈k . a,b∈N . a<c . b=c :Ↄ: Ts. (7)

c∈k . a,b∈N . a=c . b=c :Ↄ: ϶D(a,b) = ϶Dc = ϶DM϶Dc = ϶DM϶D(a,b) :Ↄ: Ts. (8)

(4)(5)(6)(7)(8).Ↄ. c∈k . a,b∈N . a<c+1 . b<c+1 : Ↄ: Ts. (9)

(9)Ↄ. c∈k .Ↄ. (c+1)∈k. (10)

(1)(10).Ↄ ∴ c∈N . Hp. a<c . b<c :Ↄ: Ts. (11)

(a+b)[c](11).Ↄ: Hp. Ↄ. Ts. (Th.)

37. a,b,m∈N .Ↄ. M϶D(am,bm) = m×M϶D(a,b).

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§7. Theoremata varia. | §7. Various theorems.

1. a,b∈N . a^2+b^2ꓷ7 :Ↄ: aꓷ7 . bꓷ7.

2. x∈N .Ↄ. x(x+1)ꓷ2.

3. x∈N .Ↄ. x(x+1)(x+2)ꓷ6.

4. x∈N .Ↄ. x(x+1)(2x+1)ꓷ6.

5. x∈N .Ↄ: x .π. x+1.

6. x∈N .Ↄ: 2x−1 .π. 2x+1.

7. x∈N .Ↄ. $(2x+1)^2-1$ꓷ8.

8. a∈N . a>1 :Ↄ∴ Np . ϶>1 . ϶Da : −= Λ. (Eucl. VII, 31)

9. a,b∈N ∴ b^2>a ∴ ϶Da . ϶>1 . ϶<b := Λ ::Ↄ. a∈Np.

10. a,b∈N . a∈Np . a −Db :Ↄ: aπb. (Eucl. VII, 29)

11. a,b,c∈N . aDbc . aπb :Ↄ. aDc.

12. a,b∈N . m = M϶D(a,b) :Ↄ: a/mπ . b/m.

13. a∈Np . b,c∈N . aDbc :Ↄ: aDb .∪. aDc. (Eucl. VII, 30)

14. a∈Np . b,n∈N :Ↄ: aDb^n .=. aDb. (Eucl. IX, 12)

15. a,b,c∈N . aπb . cDa :Ↄ: cπb. (Eucl. VII, 23)

16. a,b,c∈N .Ↄ ∴ aπb . aπc :=: aπbc. (Eucl. VII, 24)

17. a,b,c∈N . bπc . bDa . cDa :Ↄ. bcDa.

18. a,b,c∈N paπb :Ↄ: ϶D(ac,b) = ϶D(c,b).

19. a,b∈N .Ↄ. W϶ꓷ(a,b)∈N.

20. a,b∈N .Ↄ. W϶ꓷ(a,b) = ab/M϶D(a,b). (Eucl. VII, 34)

21. a,b,c∈N . cꓷa . cꓷb :Ↄ: cꓷW϶D(a,b). (Eucl. VII, 35)

22. x∈N . x<41 :Ↄ. $41-x+x^2$∈Np.

23. M . Np := Λ. (Eucl. IX, 20)

23. n∈Np . a∈N . a −ꓷn :Ↄ. $a^{n-1}-1$ꓷn. (Fermat)

§8. Numerorum rationes. | §8. Rational numbers.

Explicationes. | Explications.

Si p,q∈N, tunc $\frac{p}{q}$ legitur *ratio numeri p numero q*.

Signum R legitur *duorum numerorum ratio*, et indicat numeros rationales positivos.

If p,q∈N, then $\frac{p}{q}$ is read *the ratio of the number p to the number q*.

The sign R is read *ratio of two numbers*, and indicates the positive rational numbers.

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Definitiones. | Definitions.

1. $m,p,q \in N . \supset . m\frac{p}{q} = mp/q.$
2. $p,q,p',q' \in N :\supset: \frac{p}{q} = \frac{p'}{q'} .=. \therefore x \in N . x\frac{p}{q}, x\frac{p'}{q'} \in N : \supset_x . x\frac{p}{q} = x\frac{p'}{q'}$
3. $R = [x \in](x \in N . \ldots) . - = \Lambda.$
4. $p \in N . \supset . \frac{p}{1} = p.$

Theoremata. | Theorems.

5. $p,q,p',q' \in N :\supset: \frac{p}{q} = \frac{p'}{q'} .=. pq' = p'q.$ (Eucl. VII, 19)
Dem. Hp. $\frac{p}{q} = \frac{p'}{q'} :\supset: qq'\frac{p}{q} = qq'\frac{p'}{q'} \in N$. P2 (1)
$:\supset: qq'\frac{p}{q} = pq' . qq'\frac{p'}{q'} = p'q :\supset: pq' = p'q.$
Hp. $pq' = p'q :\supset: x \in N . x\frac{p}{q}, x\frac{p'}{q'} \in N :\supset_x : x\frac{p}{q} = x\frac{p'}{q'}$ (2)
(1)(2) $\supset$ Ts.
6. $m,p,q \in N . \supset . \frac{p}{q} = \frac{mp}{mq}.$ (Eucl. VII, 17)
7. $p,q \in N . m \in N . m D p . m D q : \supset : \frac{p}{q} = \frac{p/m}{q/m}.$
8. $p,q,p',q' \in N . \ldots \supset \ldots$
9. $p,q,p',q' \in N . \frac{p}{q} = \frac{p'}{q'} : \supset : \ldots$ M $\mathfrak{D}(p,q)$.
10. $p,q,p',q' \in N . \ldots$ (Eucl. VII, 21)
11. $p,q,p',q' \in N . \ldots$ (Eucl. VII, 13)
12. $p,q \in N . \supset : [m \in](m \in N . m\frac{p}{q} \in N) . - = \Lambda.$
13. $a \in R . \supset : [m \in](m \in N . ma \in N) . - = \Lambda.$

13. $p,q,p',q' \in N :\supset: (r,s,t)(r,s,t \in N . \frac{p}{q} = \frac{r}{t} . \frac{p'}{q'} = \frac{s}{t}) . - = \Lambda.$
17. $a,b \in R :\supset: (r,s,t)(r,s,t \in N . a = \frac{r}{t} . b = \frac{s}{t}) . - = \Lambda.$
14. $a,b,c \in R :\supset: (m,n,p,q)(m,n,p,q \in N . a = \frac{m}{p} . b = \frac{n}{p} . c = \frac{q}{p}) . - = \Lambda.$
15. $p,q,r \in N . \frac{p}{q} = \frac{p}{r} . \supset . q = r.$
16. $m \in N . a,b \in R . ma = mb . \supset . a = b.$
17. $a,b \in R : \supset : a = b .$
$\supset : a > b . - = \Lambda .$
$\supset : a = b . b > a . - = \Lambda .$
18. $N \supset R.$

Definitiones. | Definitions.

19. $a,b \in R : \supset : a > b . = \therefore x \in N . ax, bx \in N : \supset_x . ax > bx.$
20. $a,b \in R . \supset : b < a . = . a > b.$

Theoremata. | Theorems.

21. $p,q,r \in N . a = \frac{p}{r} . b = \frac{q}{r} : \supset : a > b . = . p > q.$
22. $p,q,p',q' \in N : \supset : \frac{p}{q} > \frac{p'}{q'} . = . pq' > p'q.$
23. $p,q,r \in N . a = \frac{p}{r} . b = \frac{q}{r} : \supset : a = b . = . p = q.$
24. $p,q,p',q' \in N : \supset : \frac{p}{q} = \frac{p'}{q'} . = . \frac{p+q}{q} = \frac{p'+q'}{q'}.$
25. $a \in R . \supset : a > a : - = \Lambda.$
26. $a \in R . \supset : R . a < a . - = \Lambda.$
27. $a,b \in R . a < b . \supset : R . a > x . x > b . - = \Lambda.$
28. $a,b \in R . \supset : a < b . a = b . - = \Lambda.$
$\supset : a > b . a = b . - = \Lambda.$
$\supset : a < b . a > b . - = \Lambda.$
$\supset : a < b . a = b . a > b : - = \Lambda.$
29. $a,b,c \in R : \supset : a < b . b < c : \supset : a < c.$
$\supset : a < b . b = c : \supset : a < c.$

Definitiones. | Definitions.

30. $a,b \in R . \supset a + b = [z \in](z \in N . ax, xb \in N : \supset_x . ax + xb = xz).$
31. $a,b \in R . \supset : b - a = [x \in](x \in R . a + x = b).$
32. $a,b \in R . \supset . ab = [z \in](z \in N . ax, xb \in N : \supset_x . (ax)(xb) = x^2 z).$
33. $a,b \in R . \supset . b/a = [x \in](x \in R . ax = b).$

Theoremata. | Theorems.

34. $p,q,r \in N . \supset . \frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}.$
35. $a,b \in R . \supset . a + b \in R.$
36. $p,q,r \in N . p > q . \supset . \frac{p}{r} - \frac{q}{r} = \frac{p-q}{r}.$
37. $a,b \in R . \supset . b - a \in R.$
38. $p,q,p',q' \in N . \supset . \frac{p}{q}\frac{p'}{q'} = \frac{pp'}{qq'}.$
39. $a,b \in R . \supset . ab \in R.$
40. $p,q,p',q' \in N . \supset . \frac{p}{q} / \frac{p'}{q'} = \frac{pq'}{p'q}.$
41. $a,b \in R . \supset . b/a \in R.$
42. $p,q \in N . \supset . \frac{p}{q} = \frac{p}{q}.$

§9. Rationalium systemata. Irrationales. | §9. The system of rationals. Irrationals.

Explicatio. | Explanation.

Si $a \in$ K R, signum T a legitur *terminus summus, vel limes summus classis a*. Supra hos novum ens relationes ac operationes tantum definimus.

If $a \in$ K R, the sign T a is read *upper boundary*, or *upper limit of the class a*. We shall define only a few relations and operations on this new entity.

Definitiones. | Definitions.

1. $a \in$ K R . $x \in$ R $. \supset : x = T a . = . \ldots \Lambda.$
2. $a \in$ K R . $b \in$ K R $: \supset : T a = T b . = \ldots$
3. $a \in$ K R $. \supset : T a > x . = \ldots$

Theorema. | Theorem.

4. $x \in \mathrm{R} . \supset :: x = \therefore \mathrm{T} : \mathrm{R} . \ni < x.$

Explicatio. | Explanation.

Signum Q legitur *quantitas*, numerosque indicat reales positivos, rationales aut irrationales, 0 et ∞ exceptis.

The sign Q is read *quantity*, and indicates the positive real numbers, rational or irrational, with the exception of 0 and ∞.

Definitiones. | Definitions.

5. $\mathrm{Q} = [x \in](a \in K\,\mathrm{R} : a - = \Lambda : \mathrm{R} \in > \mathrm{T}\,a . \min = \Lambda :$
$\mathrm{T}\,a = x \therefore - = \Lambda).$
6. $a,b \in \mathrm{Q} . \supset :: a = b . = \therefore \mathrm{R} . \ni < a := : \mathrm{R} . \ni < b.$
7. $a,b \in \mathrm{Q} . \supset :: a < b . = \therefore \mathrm{R} . \ni > a . \ni < b := - = \Lambda.$
8. $a,b \in \mathrm{Q} . \supset : b > a . = . a < b.$

Theoremata. | Theorems.

9. $a \in \mathrm{Q} . \supset \therefore \mathrm{R} . \ni < a : - = \Lambda.$
10. $a \in \mathrm{Q} . \supset \therefore \mathrm{R} . \ni > a : - = \Lambda.$
11. $\mathrm{R} \supset \mathrm{Q}.$

Subsistunt quoque propositiones quae a P17, 28, 29 in §8 obtinentur, si loco R legatur Q.

The propositions obtained from P17, 28, 29 in §8 also hold, by reading Q for R.

Definitiones. | Definitions.

12. $a,b \in Q . \supset . a + b = T[z \in]([(x,y) \in] : x,y \in$
$\mathrm{R} . x < a . y < b . x + y = z \therefore - = \Lambda).$
13. $a,b \in Q . \supset . ab = T[z \in]([x,y) \in] : x,y \in$
$\mathrm{R} . x < a . y < b . xy = z \therefore - = \Lambda).$

Ut valeant hae definitiones, demonstrandum est subsistere propositiones 12 et 13, si $a,b \in \mathrm{R}$.

In order for these definitions to have meaning, it must be proved that propositions 12 and 13 hold, if $a,b \in \mathrm{R}$.

Substractionem et divisionem ut operationes inversas additiones et multiplicationis definire licet, illarumque proprietas demonstrare.

Subtraction and division could be defined as the inverse operations to addition and multiplication, and their properties could be proved.

§10. Quantitatum systemata. | §10. System of quantities.

Explicationes. | Explanations.

Si $a \in$ K Q, signa **I** a, **E** a, **L** a leguntur: *interior, exterior, limes classis a*.

If $a \in$ K Q, the signs **I** a, **E** a, **L** a are read: *interior, exterior, limit of the class a*.

Definitiones. | Definitions.

1. $a \in \mathrm{K\,Q} . \supset \mathbf{I}\,a = \mathrm{Q}[x \in]([u,v) \in] :: u,v \in$
$\mathrm{Q} \therefore u < x < v \therefore \ni > u . \ni < v : \supset : a :.: - = \Lambda).$
2. $a \in \mathrm{K\,Q} . \supset . \mathbf{E}\,a = \mathbf{I}(-a).$
3. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,a = (-\mathbf{I}\,a)(-\mathbf{E}\,a).$

Theoremata. | Theorems.

4. $a \in \mathrm{K\,Q} . x,u,v \in Q . u < x < v .(\ni > u . \ni < v : \supset a) :$
$\supset . x \in \mathbf{I}\,a.$
5. $a \in \mathrm{K\,Q} . x \in \mathbf{I}\,a : \supset : [(u,v) \in](u,v \in$
$\mathrm{Q} \therefore u < x < v \therefore \ni > u . \ni < v : \supset : a) - = \Lambda.$
Dem. P1 = (P4)(P5).
6. $a \in \mathrm{K\,Q} . u,v \in \mathrm{Q} .(\ni > u . \ni < v :$
$\supset a) \therefore \supset \therefore \ni > u . \ni < v : \supset \mathbf{I}\,a.$
Dem. P6 = P4.
7. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}\,a \supset a.$
8. $a \in \mathrm{K\,Q} . \supset . \mathbf{II}\,a = \mathbf{I}\,a.$
Dem. Hp. $(\mathbf{I}\,a)[a]$ P7 $: \supset . \mathbf{II}\,a \supset \mathbf{I}\,a$ (1)
Hp. $x,u,v \in \mathrm{Q} . u < x < v .(\ni > u . \ni < v : \supset a)$. P6 (2)
$: \supset : u,v \in \mathrm{Q} . u < x < v .(\ni > u . \ni < v : \supset \mathbf{I}\,a)$
Hp. $x \in \mathbf{I}\,a .(2) : \supset : x \in \mathbf{II}\,a$ (3)
Hp. $(3) : \supset : \mathbf{I}\,a\,\mathbf{II}\,a$ (4)
Hp. $(1).(4) : \supset :$ Ts. (Theor.)
9. $a,b \in \mathrm{K\,Q} . a \supset b : \supset . \mathbf{I}\,a \supset \mathbf{I}\,b$
Dem. Hp. $x,u,v \in \mathrm{Q} . u < x < v .(\ni > u . \ni < v : \supset a) :$ (1)
$\supset \therefore \ni > u . \ni < v : \supset b$
Hp. $x \in \mathbf{I}\,a : \supset : x \in \mathbf{I}\,b$ (Theor.)
10. $a,b \in \mathrm{K\,Q} : \supset : \mathbf{I}(ab) \supset \mathbf{I}\,a$
Dem. $(ab,a)[a,b]$ P9 . = . P10
11. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}(ab) \supset (\mathbf{I}\,a)(\mathbf{I}\,b)$
Dem. P11 =: P10 .∩.$(b,a)[a,b]$ P10
12. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}\,a \supset \mathbf{I}(a \cup b)$
13. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}\,a \cup \mathbf{I}\,b \supset \mathbf{I}(a \cup b)$
14. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}(ab) = (\mathbf{I}\,a)(\mathbf{I}\,b)$
Dem. Hp. P11 $: \supset . \mathbf{I}(ab) \supset (\mathbf{I}\,a)(\mathbf{I}\,b)$ (1)

Hp. $x \in \mathrm{Q} . u,v \in \mathrm{Q} . u < x < v .(\ni > u . \ni < v :$ (2)
$\supset a) . u',v' \in \mathrm{Q} . u' < x < v' .(\ni > u' . \ni < v' :$
$\supset b) . u'' = M(u \cup u') . v'' = W(v,v') : \supset : u'',v'' \in$
$Q . u'' < x < v'' .(\ni > u'' . \ni > v'' : \supset : ab)$
Hp. $x \in \mathbf{I}\,a . x \in \mathbf{I}\,b .(2) : \supset . x \in \mathbf{I}(ab)$ (3)
Hp. $(3) : \supset : (\mathbf{I}\,a)(\mathbf{I}\,b) \supset \mathbf{I}(ab)$ (4)
Hp. (1)ă.(4) $: \supset .$ Ts.
15. $a \in \mathrm{K\,Q} . \supset . \mathbf{E}\,a \supset -a$
Dem. P15 = $(-a)[a]$ P7
16. $a \in \mathrm{K\,Q} . \supset \therefore \mathbf{I}\,a . \mathbf{E}\,a := \Lambda$
Dem. Hp. P7 . P15 $: \supset \therefore \mathbf{I}\,a . \mathbf{E}\,a : \supset : a - a := \Lambda$
17. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}\,\mathbf{E}\,a = \mathbf{E}\,a$
Dem. P17 = $(-a)[a]$ P8
18. $a,b \in \mathrm{K\,Q} . b \supset a : \supset . \mathbf{E}\,a \supset \mathbf{E}\,b$
Dem. P18 = $(-a,-b)[a,b]$ P9
19. $a,b \in \mathrm{K\,Q} . \supset : \mathbf{E}\,a \cup \mathbf{E}\,b . \supset \mathbf{E}(ab)$
20. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{E}(a \cup b) = (\mathbf{E}\,a)(\mathbf{E}\,b)$
Dem. P20 = $(-a,-b)[a,b]$ P14
21. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}(-a) = \mathbf{L}\,a$
22. $a \in \mathrm{K\,Q} . \supset \therefore \mathbf{I}\,a . \mathbf{L}\,a := \Lambda$
$\supset \therefore \mathbf{E}\,a . \mathbf{L}\,a := \Lambda$
$\supset \therefore -\mathbf{I}\,a . -\mathbf{E}\,a . -\mathbf{L}\,a := \Lambda$
Dem. P22 = P3
23. $a \in \mathrm{K\,Q} . \supset : a \supset . \mathbf{I}\,a \cup \mathbf{L}\,a$
24. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}(a\,\mathbf{L}\,a) = \Lambda$
Dem. Hp. P14 . P7 . P22
$: \supset : \mathbf{I}(a\,\mathbf{L}\,a) . = . \mathbf{I}\,a\,\mathbf{I}\,\mathbf{L}\,a . \supset . \mathbf{I}\,a\,\mathbf{L}\,a . = . \Lambda$
25. $a,b \in \mathrm{K\,Q} . a \supset b : \supset : \mathbf{L}\,a . \supset . \mathbf{I}\,b \cup \mathbf{L}\,b$
Dem. Hp. P18 $: \supset : \mathbf{E}\,b \supset \mathbf{E}\,a : \supset : \mathbf{I}\,a \cup \mathbf{L}\,a . \supset . \mathbf{I}\,b \cup \mathbf{L}\,b : \supset .$
Ts.
26. $a,b \in \mathrm{K\,Q} . \supset : \mathbf{L}(ab) \supset . \mathbf{I}\,a\,\mathbf{L}\,b \cup \mathbf{I}\,b\,\mathbf{L}\,a \cup \mathbf{L}\,a\,\mathbf{L}\,b$
Dem. Hp. $\supset : ab \supset a . ab \supset b$. P25
$: \supset : \mathbf{L}(ab) \supset \mathbf{I}\,a \cup \mathbf{L}\,a . \mathbf{L}(ab) \supset \mathbf{I}\,b \cup \mathbf{L}\,b : \supset :$
$\mathbf{L}(ab) \supset (\mathbf{I}\,a \cup \mathbf{L}\,a)(\mathbf{I}\,b \cup \mathbf{L}\,b) . \mathbf{L}(ab)(\mathbf{I}\,a)(\mathbf{I}\,b) =$
$\mathbf{L}(ab)\,\mathbf{I}(ab) = \Lambda : \supset :$ Ts.
26’ $a,b \in \mathrm{K\,Q} . \supset . \mathbf{L}(ab) \supset \mathbf{L}\,a \cup \mathbf{L}\,b$
27. $a,b \in \mathrm{K\,Q} . \supset : \mathbf{L}(a \cup b) = \mathbf{L}\,a\,\mathbf{E}\,b \cup \mathbf{L}\,b\,\mathbf{E}\,a \cup \mathbf{L}\,a\,\mathbf{L}\,b$
Dem. P27 = $(-a,-b)[a,b]$ P26

27’. $a,b \in \mathrm{K\,Q} . \supset : \mathbf{L}(a \cup b) \supset \mathbf{L}\,a \cup \mathbf{L}\,b$
28. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,\mathbf{I}\,a \supset \mathbf{L}\,a$
Dem. Hp. P7 $: \supset : \mathbf{I}\,a \supset a$. P25 $: \supset : \mathbf{L}\,\mathbf{I}\,a \supset \mathbf{I}\,a \cup \mathbf{L}\,a$ (1)
Hp. P8 . P22 $: \supset . \mathbf{L}\,\mathbf{I}\,a\,\mathbf{I}\,a = \mathbf{L}\,\mathbf{I}\,a\,\mathbf{II}\,a = \Lambda$ (2)
(1)(2)$. \supset .$ Theor.
28’. $a \in \mathrm{K\,Q} . \supset \mathbf{L}\,\mathbf{E}\,a \supset \mathbf{L}\,a$
29. $a \in \mathrm{K\,Q} . \supset . \mathbf{LL}\,a \supset \mathbf{L}\,\mathbf{I}\,a \cup \mathbf{L}\,\mathbf{E}\,a$
Dem. Hp. œ$C : \mathbf{LL}\,a = \mathbf{L}(\mathbf{I}\,a \cup \mathbf{E}\,a)$. P27’ $: \supset .$ Ts.
29’. $a \in \mathrm{K\,Q} . \supset . \mathbf{LL}\,a \supset \mathbf{L}\,a$
Dem. P29 . P28 . P28’ $: \supset .$ Theor.
30. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,a = \mathbf{I}\,\mathbf{L}\,a \cup \mathbf{LL}\,a$
Dem. Hp. P23 $: \supset . \mathbf{L}\,a \supset \mathbf{I}\,\mathbf{L}\,a \cup \mathbf{LL}\,a$ (1)
Hp. P7 $: \supset . \mathbf{I}\,\mathbf{L}\,a \supset \mathbf{L}\,a$ (2)
Hp. P29’ $: \supset . \mathbf{LL}\,a \supset \mathbf{L}\,a$ (3)
(1)(2)(3)$. \supset .$ Theor.
31. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,\mathbf{I}\,\mathbf{L}\,a \supset \mathbf{LL}\,a$ă
Dem. P31 = $(\mathbf{L}\,a)[a]$ P28
32. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}\,\mathbf{LL}\,a = \Lambda$
Dem. Hp. P29’ $: \supset : \mathbf{LL}\,a = \mathbf{L}\,a\,\mathbf{LL}\,a .(\mathbf{L}\,a)[a]$ P24 $: \supset$ Ts.
33. $a \in \mathrm{K\,Q} . \supset : \mathbf{I}\,\mathbf{L}\,\mathbf{I}\,\mathbf{L}\,a = \Lambda$
Dem. P31 . P32 $: \supset . P33$
34. $a \in \mathrm{K\,Q} . \supset . \mathbf{LL}\,\mathbf{L}\,a = \mathbf{LL}\,a$
Dem. $(\mathbf{L}\,a)[a]$ P30 /p P32 $: \supset .$ Theor.
35. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}\,a\,\mathbf{L}\,b \supset \mathbf{L}(ab)$
Dem. Hp. P14 $: \supset . \mathbf{I}\,a\,\mathbf{L}\,b\,\mathbf{I}(ab) = \mathbf{I}\,a\,\mathbf{I}\,b\,\mathbf{L}\,b = \Lambda$ (1) (1)
Hp. P2 . P14 (2) (2)
$: \supset . \mathbf{I}\,a\,\mathbf{L}\,b\,\mathbf{E}(ab) = \mathbf{I}\,a\,\mathbf{L}\,b\,\mathbf{I}(-a \cup -b) =$
$\mathbf{I}(a - b)\,\mathbf{L}\,b = \mathbf{I}\,a\,\mathbf{E}\,b\,\mathbf{L}\,b = \Lambda$
(1)(2) $\supset$ Theor.
36. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}\,a\,\mathbf{L}\,b \cup \mathbf{I}\,b\,\mathbf{L}\,a \supset \mathbf{L}\,ab.$ (Vide P26) (Cf. p. 26)
Dem. P36 =: P35 .$(b,a)[a,b]$ P35
37. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{E}\,a\,\mathbf{L}\,b \cup \mathbf{E}\,b\,\mathbf{L}\,a \cup \mathbf{L}(a \cup b).$ (Vide P27) (Cf. p. 27)
Dem. P37 = $(-a,-b)[a,b]$ P36
38. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}(a \cup b) \supset \mathbf{I}\,a \cup \mathbf{I}\,b \cup \mathbf{L}\,a\,\mathbf{L}\,b$ (Vide P13) (Cf. p. 13)
Dem. Hp. $\supset . \mathbf{I}(a \cup b) \supset (\mathbf{I}\,a \cup \mathbf{L}\,a \cup \mathbf{E}\,a)(\mathbf{I}\,b \cup \mathbf{L}\,b \cup \mathbf{E}\,b)$ (1) (1)
Hp. P20 . P16 (2) (2)
$: \supset . \mathbf{I}(a \cup b)\,\mathbf{E}\,a\,\mathbf{E}\,b = \mathbf{I}(a \cup b)\,\mathbf{E}(a \cup b) = \Lambda$

Hp. P37$: \supset : \mathbf{I}(a \cup b)$ă$(\mathbf{E}\,a\,\mathbf{L}\,b \cup \mathbf{E}\,b\,\mathbf{L}\,a) . \supset . \mathbf{I}(a \cup$ (3) (3)
$b)\,\mathbf{L}(a \cup b) . = \Lambda$
(1)(2)(3)$. \supset .$ Theor.
38’. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{E}(ab) \supset \mathbf{E}\,a \cup \mathbf{E}\,b \cup \mathbf{L}\,a\,\mathbf{L}\,b$ (Vide P19) (Cf. p. 19)
39. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}\,\mathbf{L}\,a\,\mathbf{L}\,\mathbf{I}\,a = \Lambda$
Dem. Hp. P36 $: \supset : \mathbf{I}\,\mathbf{L}\,a\,\mathbf{L}\,\mathbf{I}\,a \supset \mathbf{L}(\mathbf{L}\,a\,\mathbf{I}\,a) = \Lambda$
40. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,\mathbf{I}\,a \supset \mathbf{LL}\,a$
Dem. Hp. P28 . P30 . P39 $: \supset$ Theor.
40’. $a \in \mathrm{K\,Q} . \supset . \mathbf{L}\,\mathbf{E}\,a \supset \mathbf{LL}\,a$
41. $a \in \mathrm{K\,Q} . \supset \mathbf{LL}\,a = \mathbf{L}\,\mathbf{I}\,a \cup \mathbf{L}\,\mathbf{E}\,a$
Dem. P29 . P40 . P40’ $: \supset .$ Theor.
42. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}\,\mathbf{L}\,\mathbf{I}\,a = \Lambda$
$\supset . \mathbf{I}\,\mathbf{L}\,\mathbf{E}\,a = \Lambda$
$\supset . \mathbf{LL}\,\mathbf{I}\,a = \mathbf{L}\,\mathbf{I}\,a$
$\supset . \mathbf{LL}\,\mathbf{E}\,a = \mathbf{L}\,\mathbf{E}\,a$
43. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}(\mathbf{I}\,a \cup \mathbf{I}\,b) = \mathbf{I}\,a \cup \mathbf{I}\,b$
Dem. Hp. P7 $: \supset . \mathbf{I}(\mathbf{I}\,a \cup \mathbf{I}\,b) \supset \mathbf{I}\,a \cup \mathbf{I}\,b$ (1) (1)
Hp. P8 . P13 (2) (2)
$: \supset : \mathbf{I}\,a \cup \mathbf{I}\,b . = . \mathbf{II}\,a \cup \mathbf{II}\,b . \supset . \mathbf{I}(\mathbf{I}\,a \cup \mathbf{I}\,b)$
(1)(2) $\supset$ Theor.
44. $a,b \in \mathrm{K\,Q} . \supset . \mathbf{I}(\mathbf{LL}\,a \cup \mathbf{LL}\,b) = \Lambda$
Dem. Hp. P38 . P32 . P34 (1) (1)
$: \supset . \mathbf{I}(\mathbf{LL}\,a \cup \mathbf{LL}\,b) \supset \mathbf{LL}\,a\,\mathbf{LL}\,b \supset \mathbf{LL}\,a$
Hp. (1). P8 $: \supset . \mathbf{I}(\mathbf{LL}\,a \cup \mathbf{LL}\,b) \cup \mathbf{I}\,\mathbf{LL}\,a = \Lambda$
45. $a \in \mathrm{K\,Q} . \supset . \mathbf{I}(\mathbf{I}\,a \cup \mathbf{E}\,a) = \mathbf{I}\,a \cup \mathbf{E}\,a$
Dem. P8 . P17 .$(-a)[b]$ P43 $: \supset .$ Theor.

45'. $a \in K\, Q\, .\supset .\, \mathbf{E}\,\mathbf{L}\,a = \mathbf{I}\,a \cup \mathbf{E}\,a$

46. $a \in K\, Q\, .\supset .\, \mathbf{E}\,\mathbf{I}\,a = -(\mathbf{I}\,a \cup \mathbf{L}\,\mathbf{I}\,a)$

46'. $a \in K\, Q\, .\supset .\, \mathbf{E}\mathbf{E}\,a = -(\mathbf{E}\,a \cup \mathbf{L}\,\mathbf{E}\,a)$ 46'.

Footnotes

1 H. Kennedy, *Peano. Life and Works of Giuseppe Peano*, San Francisco: Peremptory Publications, 2002, p. 41.

2 G. Peano, *Arithmetices principia, nova methodo exposita*, Bocca, Torino, 1889.

3 These English translations listed are the only ones (to my knowledge) and all the English translations listed in:

I. Grattan-Guiness (ed.), *Landmark Writings in Western Mathematics 1640-1940*, Amsterdam: Elsevier, 2005, p. 614.

4 G. Peano, (1889), "The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 83-97.

5 G. Peano, *Selected works of Giuseppe Peano*, H. Kennedy (ed.), London: George Allen & Unwin, 1973, p. 101-134.

6 Written by Vincent Verheyen. Last updated on 17/8/2015. I encourage you to use your reason for good. If you want my support, please contact me via http://vincentverheyen.com/contact. It is possible to contribute to the flourishing of knowledge, even when you have an intelligence like mine. Thank you and good luck studying.

I would like to thank Mauro Allegranzo and acknowledge his support of this work and his various comments during its creation.

7 **Giuseppe Peano's footnote (original):**

Boole:

The mathematical analysis of logic ..., Cambridge, 1847.

The calculus of logic, Camb. and Dublin Math. Journal, 1848.

An investigation of the laws of thought ..., London, 1854.

E. Schröder:

Der Operationskreis des Logikkalculus, Leipzig, 1877.

Ipse iam nonnulla quae ad logicam pertinent tractavit in praecedenti opera.

Lehrbuch der Arithmetik und Algebra ..., Leipzig, 1873.

Boole e Schröder theorias brevissime exposui in meo libro *Calcolo geometrico* ..., Torino, 1888.

Vide:

C. S. Pierce, *On the Algebra of logic*; American Journal, III, 15; VII, 180.

Jevons, *The principles of science*, London, 1883.

Mc.Coll., *The calculus of equivalent statements*, Proceedings of the London Math. Society, 1878, Vol. IX, 9. Vol X, 16.

7 **Giuseppe Peano's footnote (translated):**

Boole:

The mathematical analysis of logic ... (Cambridge, 1847.)

'The calculus of logic,' *Camb. and Dublin Math. J.*, 3 (1848), 193-98.

An investigation of the laws of thought ... (London, 1854).

E. Schröder:

Der Operationskreis des Logikkalculus (Leipzig, 1877).

He had already treated several matters pertaining to logic in a preceding work.

Lehrbuch der Arithmetik und Algebra ... (Leipzig, 1873).

I gave a very brief presentation of the theories of Boole and Schröder in my book *Calcolo geometric* etc. (Torino, 1888).

Cf:

C. S. Pierce, 'On the Algebra of logic,' *American J. Math.*, 3 (1880), 15-57; 7 (1885), 180-202.

Jevons, *The principles of science* (London, 1883).

Mc.Coll., 'The calculus of equivalent statements,' *Proc. London Math. Soc.*, 9 (1878), 9-20; 10 (1878), 16-28.

8 The 2 other translation, mentioned at the beginning of this current document, translated "ähnlich" literally to "similar", instead of "equivalent". However, additional information can be found in a footnote of the first translation:

"Today "similar" has another meaning and instead we would say "equivalent"."

G. Peano, (1889), "The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 93.