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The number of different orders to execute associative operations for n elements & 1 operator type

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In 11/2014, I started reading a mathematics book entitled "Calculus" by Michael Spivak, which is (in my biased since under-experienced view) a more accessible manual, a clear introductory expansion of some key notions of the subject. It was perhaps this lucidity that made me curious about the number of possibilities noted in the following passage (which starts on the first page of the prologue). Michael Spivak writes:

> It might seem reasonable to regard addition as an operation which can be performed on several numbers at once, and consider the sum $a_1 + ... + a_n$ of n numbers $a_1, ..., a_n$ as a basic concept. It is more convenient, however, to consider addition of pairs of numbers only, and to define other sums in terms of sums of this type. For the sum of three numbers a, b, and c, this may be done in two different ways. One can first add b and c, obtaining $b + c$, and then add a to this number, obtaining $a + (b + c)$; or one can first add a and b, and then add the sum $a + b$ to c, obtaining $(a + b) + c$. Of course, the two compound sums obtained are equal, and this fact is the very first property we shall list:
>
> (P1) If a, b, and c are any numbers, then
> $$a + (b + c) = (a + b) + c.$$
>
> The statement of this property clearly renders a separate concept of the sum of three numbers superfluous; we simply agree that $a+b+c$ denotes the number $a + (b + c) = (a + b) + c$. Addition of four numbers requires similar, though slightly more involved, considerations. The symbol $a + b + c + d$ is defined to mean
>
> (1) $((a + b) + c) + d$,
> or (2) $(a + (b + c)) + d$,
> or (3) $a + ((b + c) + d)$,
> or (4) $a + (b + (c + d))$,
> or (5) $(a + b) + (c + d)$.
>
> This definition is unambiguous since these numbers are all equal. Fortunately, *this* fact need not be listed separately, since it follows from the property P1 already listed. For example, we know from P1 that
> $$(a + b) + c = a + (b + c),$$
> and it follows immediately that (1) and (2) are equal. The equality of (2) and (3) is a direct consequence of P1, although this may not be apparent at first sight (one must let $b + c$ play the role of b in P1, and d the role of c). The equalities (3) = (4) = (5) are also simple to prove. It is probably obvious that an appeal to P1 will also suffice to prove the equality of the 14 possible ways of summing five numbers, but it may not be so clear how we can reasonably arrange a proof that this is so without actually listing these 14 sums. Such a procedure is feasible, but would soon cease to be if we considered collections of six, seven, or more numbers; it would be totally inadequate to prove the equality of all possible sums of an arbitrary finite collection of numbers $a_1, ..., a_n$.

In the passage above, Spivak first yields 2 possibilities for $n = 3$, then 5 possibilities for $n = 4$, and subsequently 14 possibilities for $n = 5$. This current article has been written since I wanted to better understand the reasons for this sequence, and subsequently invite anyone to discover an implicit &/or explicit formula which could calculated these possibilities for any given n. Of course, it is necessary (at least for someone with my very limited intelligence) to first know the quantity of these possibilities for at least some n.

It also has to be noted that, even when one follows the convention that operations inside the most brackets take precedence in order of execution, the inclusion or exclusion of some orders was not intuitively clear. E.g. even for $n = 4$, one might argue that the order of execution in the following formula is not unambiguous:

$$(a + b) + (c + d)$$

One might argue that this formula gives rise to 2 different orders to execute the operations, namely:

$$((a + b)) + (c + d)$$
$$(a + b) + ((c + d))$$

Similar objections, but for many more formulas, could be made for $n \geq 5$. We have, however chosen the same axioms as Spivak, which do not prefer these objections. Perhaps that re-calculating the possibilities which do take these objections into account, would give rise to an implicit &/or explicit formula; both for the augmented possibilities which would arise from such objections, as well as for the possibilities described below which arise from our more limitative axioms of inclusion (given e.g. a "correction" factor for these limitations, dependent of n)?

But then, how to define our limitative axioms, used by both Spivak & currently by ourselves? To answer this question, let's go back to the formulas (for $n = 4$) exampled above. We have chosen that the execution of operations of the first of the 3 formulas is unambiguously ordered, since we have chosen an axiom to be that the order of addition does not need to be specified as long as any given operation adds elements which are on the same "depth" of parentheses. Formulated in a more positive manner: one can view this axiom as the simultaneous execution of operations on the same "depth" of parentheses. Combined with this axiom, goes the axiom to limit the possible number of orders as much as possible, though unambiguously.

The following table represents the amount of possibilities (second column) for n elements (first column).

n	# (possibilities)
1	0
2	1
3	2
4	5
5	14
6	42
7	132
8	429
9	1,430
10	4,862
11	16,796
12	58,786
13	208,012
14	742,900
15	2,674,440
...	...

The number of possibilities (second column) can be found calculated in the pages below. Starting from $n = 7$ (and a fortiori from $n = 9$), the calculations have been noted in a more concise manner.

In fact, it can be noted that the algorithm used to calculate the possibilities below, is an implicit formula itself. This formula, then, would be described something like such ($n = 0$ can yield but 0 possibilities, since there is no operator):

For any $n \in \mathbb{N}_{>1}$ for which you would like to calculate the possibilities:

Find all pairs of addends $\in \mathbb{N}_{>0}$, which, when summed in pair, yield n.

Then (except when one of the addends $= 1$, then you multiply by 1, instead of 0, for this addend):

Multiply the amount of possibilities of the addends, when $\text{addend}_1 = \text{addend}_2$.

Multiply the amount of possibilities of the addends, when $\text{addend}_1 \neq \text{addend}_2$, and further multiply by a factor 2.

Lastly, sum all of the possibilities (the products, acquired from the multiplications above) for each pair (of addends).

Perhaps though, there is another or even easier implicit formula to find? Further more, we have not been able to discover any explicit formula yet.

Below, a second approach with less restrictive axioms will be used which does take the objections (noted above) into account. But first, as announced, here follow the first calculations of the possibilities, namely for $n \leq 15$.

$n = 1$ yields 0 possibilities:

$$a \tag{0}$$

$n = 2$ yields 1 possibility:

$$(a + b) \tag{1}$$

$n = 3$ yields 2 possibilities:

3 elements can be divided into groups of 2 & 1. Each group of 2 yields 1 possibility (cf. $n = 2$):

$$(a + b) \quad + \quad c \tag{1}$$

$$a \quad + \quad (b + c) \tag{2}$$

$n = 4$ yields 5 possibilities:

4 elements can be divided into groups of 3 & 1. Each group of 3 yields 2 possibilities (cf. $n = 3$):

$$((a+b)+c) \quad + \quad d \tag{1}$$

$$(a+(b+c)) \quad + \quad d \tag{2}$$

$$a \quad + \quad ((b+c)+d) \tag{3}$$

$$a \quad + \quad (b+(c+d)) \tag{4}$$

4 elements can also be divided into groups of 2 & 2. Each group of 2 yields 1 possibility (cf. $n = 2$):

$$(a+b) \quad + \quad (c+d) \tag{5}$$

$n = 5$ yields 14 possibilities:

5 elements can be divided into groups of 4 & 1. Each group of 4 yields 5 possibilities (cf. $n = 4$):

$$(((a+b)+c)+d) \quad + \quad e \tag{1}$$
$$((a+(b+c))+d) \quad + \quad e \tag{2}$$
$$(a+((b+c)+d)) \quad + \quad e \tag{3}$$
$$(a+(b+(c+d))) \quad + \quad e \tag{4}$$
$$((a+b)+(c+d)) \quad + \quad e \tag{5}$$
$$a \quad + \quad (((b+c)+d)+e) \tag{6}$$
$$a \quad + \quad ((b+(c+d))+e) \tag{7}$$
$$a \quad + \quad (b+((c+d)+e)) \tag{8}$$
$$a \quad + \quad (b+(c+(d+e))) \tag{9}$$
$$a \quad + \quad ((b+c)+(d+e)) \tag{10}$$

5 elements can also be divided into groups of 3 & 2. Each group of 3 yields 2 possibilities (cf. $n = 3$):

$$((a+b)+c) \quad + \quad (d+e) \tag{11}$$
$$(a+(b+c)) \quad + \quad (d+e) \tag{12}$$
$$(a+b) \quad + \quad ((c+d)+e) \tag{13}$$
$$(a+b) \quad + \quad (c+(d+e)) \tag{14}$$

$n = 6$ yields 42 possibilities:

6 elements can be divided into groups of 5 & 1. Each group of 5 yields 14 possibilities (cf. $n = 5$):

$$(((a+b)+c)+(d+e)) \quad + \quad f \tag{1}$$
$$((a+(b+c))+(d+e)) \quad + \quad f \tag{2}$$
$$((a+b)+((c+d)+e)) \quad + \quad f \tag{3}$$
$$((a+b)+(c+(d+e))) \quad + \quad f \tag{4}$$
$$((((a+b)+c)+d)+e) \quad + \quad f \tag{5}$$
$$(((a+(b+c))+d)+e) \quad + \quad f \tag{6}$$
$$((a+((b+c)+d))+e) \quad + \quad f \tag{7}$$
$$((a+(b+(c+d)))+e) \quad + \quad f \tag{8}$$
$$(((a+b)+(c+d))+e) \quad + \quad f \tag{9}$$
$$(a+(((b+c)+d)+e)) \quad + \quad f \tag{10}$$
$$(a+((b+(c+d))+e)) \quad + \quad f \tag{11}$$
$$(a+(b+((c+d)+e))) \quad + \quad f \tag{12}$$
$$(a+(b+(c+(d+e)))) \quad + \quad f \tag{13}$$
$$(a+((b+c)+(d+e))) \quad + \quad f \tag{14}$$
$$a \quad + \quad (((b+c)+d)+(e+f)) \tag{15}$$
$$a \quad + \quad ((b+(c+d))+(e+f)) \tag{16}$$
$$a \quad + \quad ((b+c)+((d+e)+f)) \tag{17}$$
$$a \quad + \quad ((b+c)+(d+(e+f))) \tag{18}$$
$$a \quad + \quad ((((b+c)+d)+e)+f) \tag{19}$$
$$a \quad + \quad (((b+(c+d))+e)+f) \tag{20}$$
$$a \quad + \quad ((b+((c+d)+e))+f) \tag{21}$$
$$a \quad + \quad ((b+(c+(d+e)))+f) \tag{22}$$
$$a \quad + \quad (((b+c)+(d+e))+f) \tag{23}$$
$$a \quad + \quad (b+(((c+d)+e)+f)) \tag{24}$$
$$a \quad + \quad (b+((c+(d+e))+f)) \tag{25}$$
$$a \quad + \quad (b+(c+((d+e)+f))) \tag{26}$$
$$a \quad + \quad (b+(c+(d+(e+f)))) \tag{27}$$
$$a \quad + \quad (b+((c+d)+(e+f))) \tag{28}$$

6 elements can also be divided into groups of 4 & 2. Each group of 4 yields 5 possibilities (cf. $n = 4$):

$$((a+b)+(c+d)) \quad + \quad (e+f) \tag{29}$$
$$(((a+b)+c)+d) \quad + \quad (e+f) \tag{30}$$
$$((a+(b+c))+d) \quad + \quad (e+f) \tag{31}$$
$$(a+((b+c)+d)) \quad + \quad (e+f) \tag{32}$$
$$(a+(b+(c+d))) \quad + \quad (e+f) \tag{33}$$
$$(a+b) \quad + \quad ((c+d)+(e+f)) \tag{34}$$
$$(a+b) \quad + \quad (((c+d)+e)+f) \tag{35}$$
$$(a+b) \quad + \quad ((c+(d+e))+f) \tag{36}$$
$$(a+b) \quad + \quad (c+((d+e)+f)) \tag{37}$$
$$(a+b) \quad + \quad (c+(d+(e+f))) \tag{38}$$

6 elements can also be divided into groups of 3 & 3. Each group of 3 yields 2 possibilities (cf. $n = 3$):

$$((a+b)+c) \quad + \quad ((d+e)+f) \tag{39}$$
$$((a+b)+c) \quad + \quad (d+(e+f)) \tag{40}$$
$$(a+(b+c)) \quad + \quad ((d+e)+f) \tag{41}$$
$$(a+(b+c)) \quad + \quad (d+(e+f)) \tag{42}$$

$n = 7$ yields 132 possibilities:

7 elements can be divided into groups of 6 & 1. Each group of 6 yields 42 possibilities (cf. $n = 6$):

(elements 1 to 6)	+	element 7	(42)
element 1	+	(elements 2 to 7)	(84)

7 elements can also be divided into groups of 5 & 2. Each group of 5 yields 14 possibilities (cf. $n = 5$):

(elements 1 to 5)	+	(elements 6 & 7)	(98)
(elements 1 & 2)	+	(elements 3 to 7)	(112)

7 elements can also be divided into groups of 4 & 3. Each group of 4 yields 5 possibilities (cf. $n = 4$). Each group of 3 yields 2 possibilities (cf. $n = 3$):

(elements 1 to 4)	+	(elements 5 to 7)	(122)
(elements 1 to 3)	+	(elements 4 to 7)	(132)

$n = 8$ yields 429 possibilities:

8 elements can be divided into groups of 7 & 1. Each group of 7 yields 132 possibilities (cf. $n = 7$):

(elements 1 to 7)	+	element 8	(132)
element 1	+	(elements 2 to 8)	(264)

8 elements can also be divided into groups of 6 & 2. Each group of 6 yields 42 possibilities (cf. $n = 6$):

(elements 1 to 6)	+	(elements 7 & 8)	(306)
(elements 1 & 2)	+	(elements 3 to 8)	(348)

8 elements can also be divided into groups of 5 & 3. Each group of 5 yields 14 possibilities (cf. $n = 5$). Each group of 3 yields 2 possibilities (cf. $n = 3$):

(elements 1 to 5)	+	(elements 6 & 8)	(376)
(elements 1 & 3)	+	(elements 4 to 8)	(404)

8 elements can also be divided into groups of 4 & 4. Each group of 4 yields 5 possibilities (cf. $n = 4$):

(elements 1 to 4)	+	(elements 5 & 8)	(429)

$n = 9$ yields 1,430 possibilities:

9 elements can be divided into groups of 8 & 1, which yields:

$$(429) \times 2 = 858 \tag{858}$$

9 elements can also be divided into groups of 7 & 2, which yields:

$$(132) \times 2 = 264 \tag{1122}$$

9 elements can also be divided into groups of 6 & 3, which yields:

$$(42 \times 2) \times 2 = 168 \tag{1290}$$

9 elements can also be divided into groups of 5 & 4, which yields:

$$(14 \times 5) \times 2 = 140 \tag{1430}$$

$n = 10$ yields 4,862 possibilities:

10 elements can be divided into groups of 9 & 1, which yields:

$$(1{,}430) \times 2 = 2{,}860 \qquad (2860)$$

10 elements can also be divided into groups of 8 & 2, which yields:

$$(429) \times 2 = 858 \qquad (3718)$$

10 elements can also be divided into groups of 7 & 3, which yields:

$$(132 \times 2) \times 2 = 528 \qquad (4246)$$

10 elements can also be divided into groups of 6 & 4, which yields:

$$(42 \times 5) \times 2 = 420 \qquad (4666)$$

10 elements can also be divided into groups of 5 & 5, which yields:

$$(14^2) = 196 \qquad (4862)$$

$n = 11$ yields 16,796 possibilities:

11 elements can be divided into groups of 10 & 1, which yields:

$$(4{,}862) \times 2 = 9{,}724 \tag{9724}$$

11 elements can also be divided into groups of 9 & 2, which yields:

$$(1{,}430) \times 2 = 2{,}860 \tag{12584}$$

11 elements can also be divided into groups of 8 & 3, which yields:

$$(429 \times 2) \times 2 = 1{,}716 \tag{14300}$$

11 elements can also be divided into groups of 7 & 4, which yields:

$$(132 \times 5) \times 2 = 1{,}320 \tag{15620}$$

11 elements can also be divided into groups of 6 & 5, which yields:

$$(42 \times 14) \times 2 = 1{,}176 \tag{16796}$$

$n = 12$ yields 58,786 possibilities:

12 elements can be divided into groups of 11 & 1, which yields:

$$(16{,}796) \times 2 = 33{,}592 \tag{33592}$$

12 elements can also be divided into groups of 10 & 2, which yields:

$$(4{,}862) \times 2 = 9{,}724 \tag{43316}$$

12 elements can also be divided into groups of 9 & 3, which yields:

$$(1{,}430 \times 2) \times 2 = 5{,}720 \tag{49036}$$

12 elements can also be divided into groups of 8 & 4, which yields:

$$(429 \times 5) \times 2 = 4{,}290 \tag{53326}$$

12 elements can also be divided into groups of 7 & 5, which yields:

$$(132 \times 14) \times 2 = 3{,}696 \tag{57022}$$

12 elements can also be divided into groups of 6 & 6, which yields:

$$(42^2) = 1{,}764 \tag{58786}$$

$n = 13$ yields 208,012 possibilities:

13 elements can be divided into groups of 12 & 1, which yields:

$$(58{,}786) \times 2 = 117{,}572 \tag{117572}$$

13 elements can also be divided into groups of 11 & 2, which yields:

$$(16{,}796) \times 2 = 33{,}592 \tag{151164}$$

13 elements can also be divided into groups of 10 & 3, which yields:

$$(4{,}862 \times 2) \times 2 = 19{,}448 \tag{170612}$$

13 elements can also be divided into groups of 9 & 4, which yields:

$$(1{,}430 \times 5) \times 2 = 14{,}300 \tag{184912}$$

13 elements can also be divided into groups of 8 & 5, which yields:

$$(429 \times 14) \times 2 = 12{,}012 \tag{196924}$$

13 elements can also be divided into groups of 7 & 6, which yields:

$$(132 \times 42) \times 2 = 11{,}088 \tag{208012}$$

$n = 14$ yields 742,900 possibilities:

14 elements can be divided into groups of 13 & 1, which yields:

$$(208{,}012) \times 2 = 416{,}024 \qquad (416024)$$

14 elements can also be divided into groups of 12 & 2, which yields:

$$(58{,}786) \times 2 = 117{,}572 \qquad (533596)$$

14 elements can also be divided into groups of 11 & 3, which yields:

$$(16{,}796 \times 2) \times 2 = 67{,}184 \qquad (600780)$$

14 elements can also be divided into groups of 10 & 4, which yields:

$$(4{,}862 \times 5) \times 2 = 48{,}620 \qquad (649400)$$

14 elements can also be divided into groups of 9 & 5, which yields:

$$(1{,}430 \times 14) \times 2 = 40{,}040 \qquad (689440)$$

14 elements can also be divided into groups of 8 & 6, which yields:

$$(429 \times 42) \times 2 = 36{,}036 \qquad (725476)$$

14 elements can also be divided into groups of 7 & 7, which yields:

$$(132^2) = 17{,}424 \qquad (742900)$$

$n = 15$ yields 2,674,440 possibilities:

15 elements can be divided into groups of 14 & 1, which yields:

$$(742{,}900) \times 2 = 1{,}485{,}800 \tag{1485800}$$

15 elements can also be divided into groups of 13 & 2, which yields:

$$(208{,}012) \times 2 = 416{,}024 \tag{1901824}$$

15 elements can also be divided into groups of 12 & 3, which yields:

$$(58{,}786 \times 2) \times 2 = 235{,}144 \tag{2136968}$$

15 elements can also be divided into groups of 11 & 4, which yields:

$$(16{,}796 \times 5) \times 2 = 167{,}960 \tag{2304928}$$

15 elements can also be divided into groups of 10 & 5, which yields:

$$(4{,}862 \times 14) \times 2 = 136{,}136 \tag{2441064}$$

15 elements can also be divided into groups of 9 & 6, which yields:

$$(1{,}430 \times 42) \times 2 = 120{,}120 \tag{2561184}$$

15 elements can also be divided into groups of 8 & 7, which yields:

$$(429 \times 132) \times 2 = 113{,}256 \tag{2674400}$$

Since we have not been able to find a simple formula to calculate the possibilities, following the axioms described above; we will now try to do to the same for completely unambiguously ordered operations, where it is now not allowed to have more than one operation on the same "depth" of parentheses.

Perhaps then, afterwards, a formula can be discovered which can satisfy the possibilities of the more limited axioms described above?

Since, below, we will need to place an unambiguous order on each operator, without any exception; it is possible to expect the standard formula for permutations as a combinatoric solution. This has also been our assumption. We will now demonstrate, for pedagogical reasons, that this is indeed the case for $n \leq 5$, and thus will first list the possibilities:

$n = 1$ yields 0 possibilities:

$$a \tag{0}$$

$n = 2$ yields 1 possibility:

$$(a + b) \tag{1}$$

$n = 3$ yields 2 possibilities:

3 elements can be divided into groups of 2 & 1. Each group of 2 yields 1 possibility (cf. $n = 2$):

$$(a+b) \quad + \quad c \tag{1}$$

$$a \quad + \quad (b+c) \tag{2}$$

$n = 4$ yields 6 possibilities:

4 elements can be divided into groups of 3 & 1. Each group of 3 yields 2 possibilities (cf. $n = 3$):

$$((a + b) + c) \quad + \quad d \tag{1}$$
$$(a + (b + c)) \quad + \quad d \tag{2}$$
$$a \quad + \quad ((b + c) + d) \tag{3}$$
$$a \quad + \quad (b + (c + d)) \tag{4}$$

4 elements can also be divided into groups of 2 & 2. Each group of 2 yields 2 possibilities (cf. $n = 2$):

$$((a + b)) \quad + \quad (c + d) \tag{5}$$
$$(a + b) \quad + \quad ((c + d)) \tag{6}$$

$n = 5$ yields 16 possibilities:

5 elements can be divided into groups of 4 & 1. Each group of 4 yields 6 possibilities (cf. $n = 4$):

$$(((a+b)+c)+d) \quad + \quad e \tag{1}$$
$$((a+(b+c))+d) \quad + \quad e \tag{2}$$
$$(a+((b+c)+d)) \quad + \quad e \tag{3}$$
$$(a+(b+(c+d))) \quad + \quad e \tag{4}$$
$$(((a+b))+(c+d)) \quad + \quad e \tag{5}$$
$$((a+b)+((c+d))) \quad + \quad e \tag{6}$$
$$a \quad + \quad (((b+c)+d)+e) \tag{7}$$
$$a \quad + \quad ((b+(c+d))+e) \tag{8}$$
$$a \quad + \quad (b+((c+d)+e)) \tag{9}$$
$$a \quad + \quad (b+(c+(d+e))) \tag{10}$$
$$a \quad + \quad (((b+c))+(d+e)) \tag{11}$$
$$a \quad + \quad ((b+c)+((d+e))) \tag{12}$$

5 elements can also be divided into groups of 3 & 2. Each group of 3 yields 2 possibilities (cf. $n = 3$):

$$((a+b)+c) \quad + \quad (d+e) \tag{13}$$
$$(a+(b+c)) \quad + \quad (d+e) \tag{14}$$
$$(a+b) \quad + \quad ((c+d)+e) \tag{15}$$
$$(a+b) \quad + \quad (c+(d+e)) \tag{16}$$